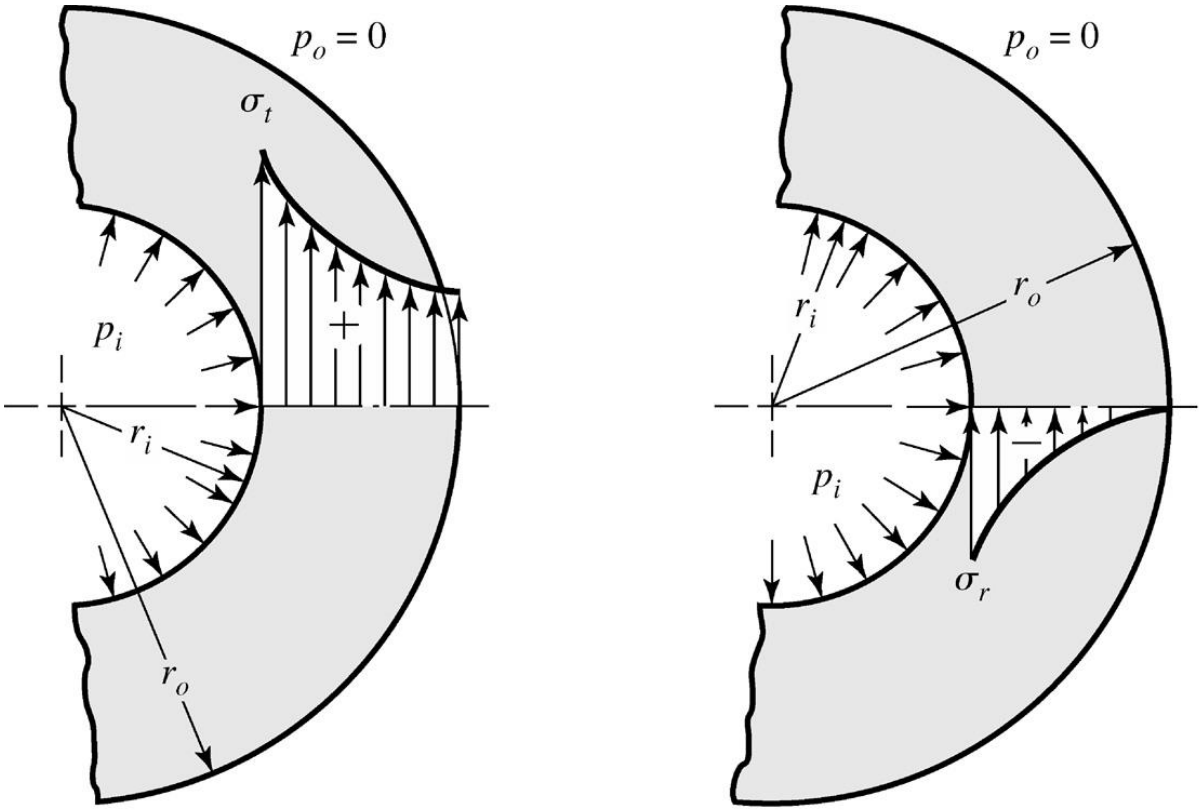
Stress Analysis II

Stress distributions in a thick walled cylinder subjected to internal pressure



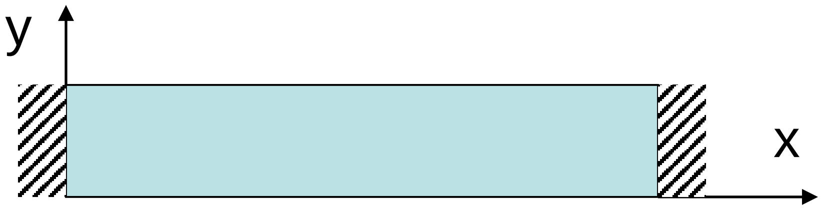
(*a*) Tangential stress distribution

(*b*) Radial stress distribution

Thermal Stresses and Strains

$\varepsilon = \varepsilon_T + \varepsilon_E$ ε_T : Thermal strain ε_E : Elastic Strain

Uniform temperature change:
if there is no constraint, there is no stress.

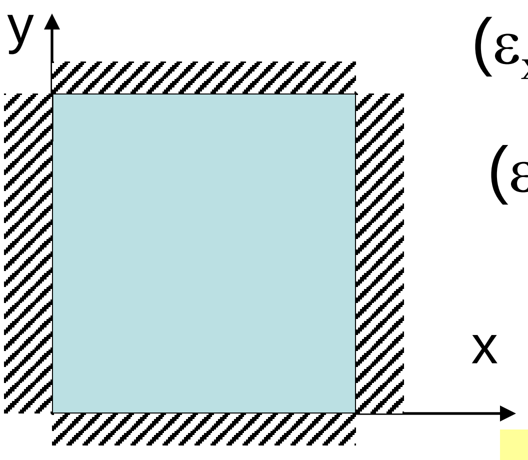


$$\varepsilon_x = (\varepsilon_x)_T + (\varepsilon_x)_E = 0$$

$$(\varepsilon_x)_E = -(\varepsilon_x)_T$$

$$\sigma_x = E\,(\varepsilon_x)_E = -E\,(\varepsilon_x)_T = -E\,\alpha\,\Delta T$$

Plane Stress:



$$(\varepsilon_x)_T = (\varepsilon_y)_T = \alpha\,\Delta T$$

$$\varepsilon_x = (\varepsilon_x)_T + (\varepsilon_x)_E = 0$$

$$(\varepsilon_x)_E = -(\varepsilon_x)_T = -\alpha\,\Delta T$$

but $(\varepsilon_y)_E = (\varepsilon_x)_E$

$$\sigma_x = \frac{E}{1-\nu^2}\left[(\varepsilon_x)_E + \nu(\varepsilon_y)_E\right]$$

$$\sigma_x = \frac{E(1+\nu)(\varepsilon_x)_E}{1-\nu^2}$$

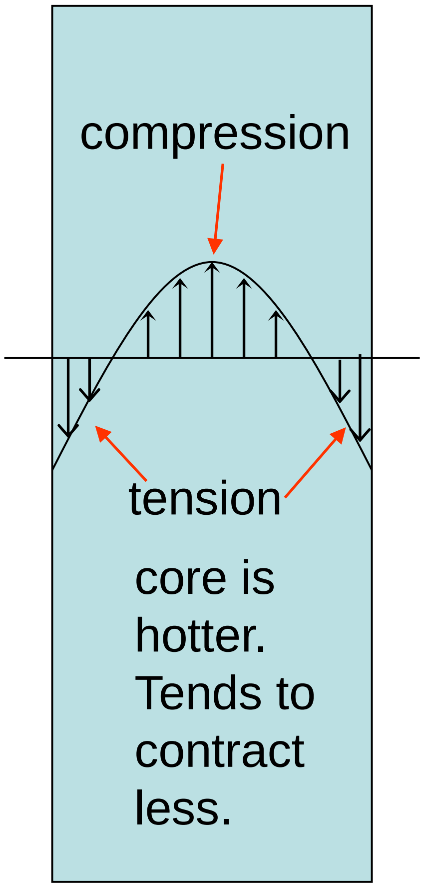
$$\sigma_x = -\frac{E\alpha\Delta T}{1-\nu}$$

temperature stress arise because of constraint.

THERMAL STRESSes arise because of temperature gradient in a member.

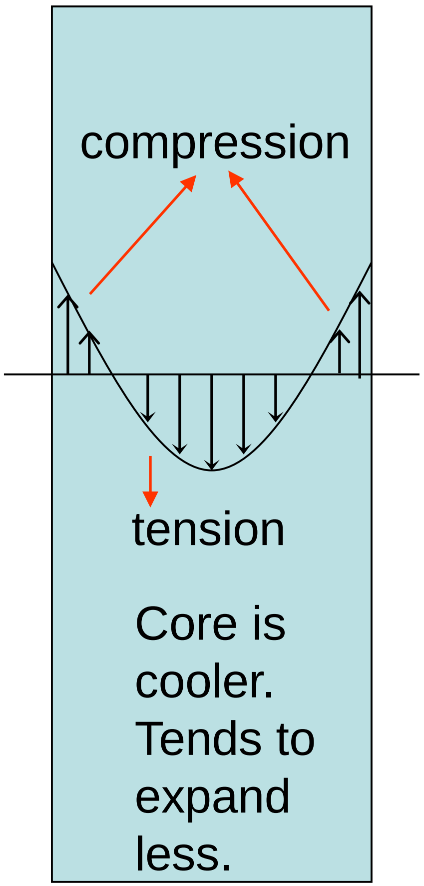
Infinite slab during heating and cooling:

surfaces are cooler. Tend to contract more.



Cooling

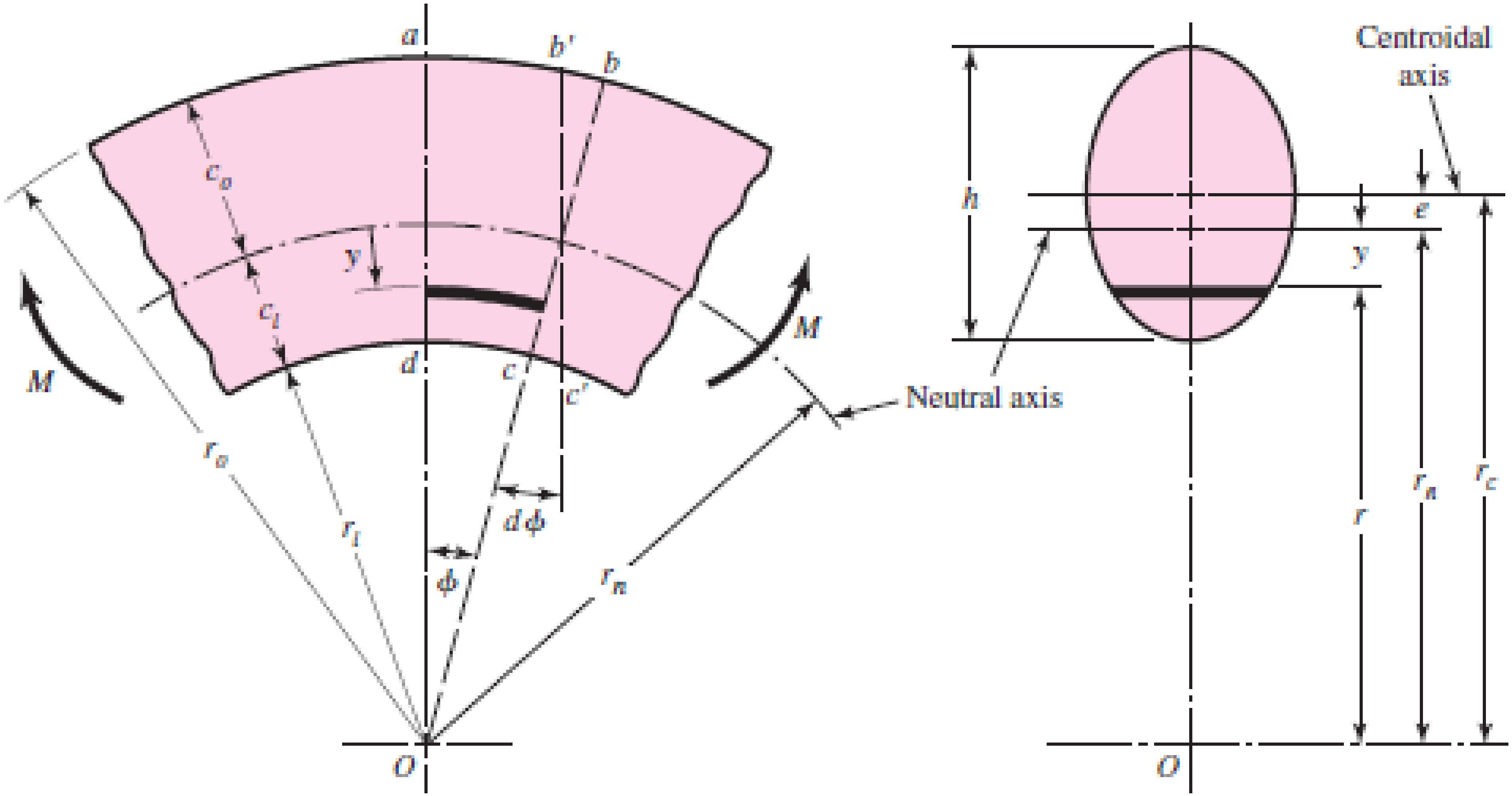
surfaces are hotter. Tend to expand more.



Heating

Curved Members in Flexure

- Distribution of stress in a curved flexural member is determined by using the following assumptions.
  - Cross-section has an axis of symmetry.
  - Plane cross-sections remain plane after bending.
  - Modulus of elasticity is the same in tension and compression



For curved beams;
neutral axis is not coincident with centroidal axis, and stress distribution is not linear.

bc rotates through dΦ to b'c'.

Strain on a fiber at r ; $\varepsilon = \dfrac{\delta}{\ell} = \dfrac{(r_n - r)d\phi}{r\phi} = \dfrac{yd\phi}{r\phi}$ ....(a)

Corresponding normal stress; $\sigma = E\varepsilon = \dfrac{E\delta}{\ell} = \dfrac{E(r_n - r)d\phi}{r\phi} = \dfrac{Eyd\phi}{r\phi}$.................(b)

There is no external force on the beam, therefore the sum of the normal forces on the cross-section of the beam must be zero.

$$\int_A \sigma\, dA = E\frac{d\phi}{\phi}\int_A \frac{(r_n - r)}{r}\, dA = 0$$ ....(c)

$$E\frac{d\phi}{\phi}\left(r_n\int_A \frac{dA}{r} - \int_A dA\right) = 0$$ ....(d)

$$r_n\int_A \frac{dA}{r} - A = 0$$ ....(e)

$$r_n = \frac{A}{\int_A \frac{dA}{r}}$$

location of neutral axis

Note that location of centroid is given by $r_c = \dfrac{\int_A r\, dA}{A}$

hence r_c and r_n are not the same.

The external moment on the beam is equal to the moment of the normal forces on the cross-section of the beam.

$$M=\int_A y\sigma\,dA=\int_A (r_n-r)\sigma\,dA=E\frac{d\phi}{\phi}\int_A\frac{(r_n-r)^2}{r}dA \qquad \text{.................(f)}$$

note that $(r_n-r)^2=r_n{}^2-2r_nr+r^2$

$$M=E\frac{d\phi}{\phi}\left(\underbrace{r_n{}^2\int_A\frac{dA}{r}-r_n\int_A dA}_{r_n\left(r_n\int_A\frac{dA}{r}-\int_A dA\right)=0}-r_n\int_A dA+\int_A r\,dA\right) \qquad \text{.................(g)}$$

$$M=E\frac{d\phi}{\phi}\left(-r_n\underbrace{\int_A dA}_{A}+\underbrace{\int_A r\,dA}_{r_cA}\right)$$

$$M=E\frac{d\phi}{\phi}(r_c-r_n)A=E\frac{d\phi}{\phi}eA$$

from (b)

$$\frac{d\phi}{\phi} = \frac{r\sigma}{E(r_n - r)}$$

$$M = \cancel{E}\frac{r\sigma}{\cancel{E}(r_n - r)}eA$$

$$\sigma = \frac{M(r_n - r)}{reA}$$

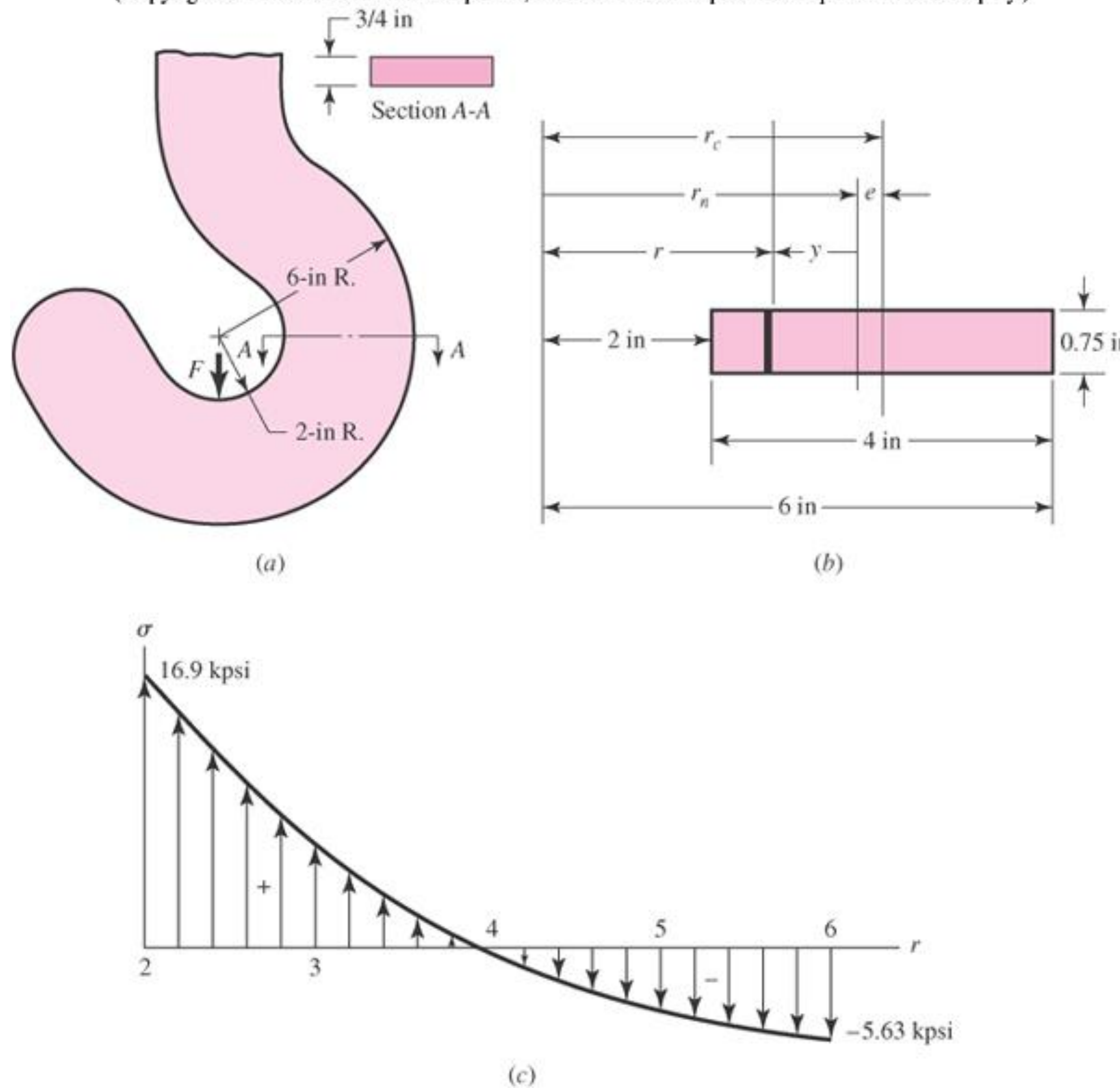
but $y = r_n - r$

$$r = r_n - y$$

$$\sigma = \frac{My}{(r_n - y)eA}$$

$$\sigma_i = \frac{Mc_i}{r_i eA} \qquad \sigma_o = -\frac{Mc_o}{r_o eA}$$

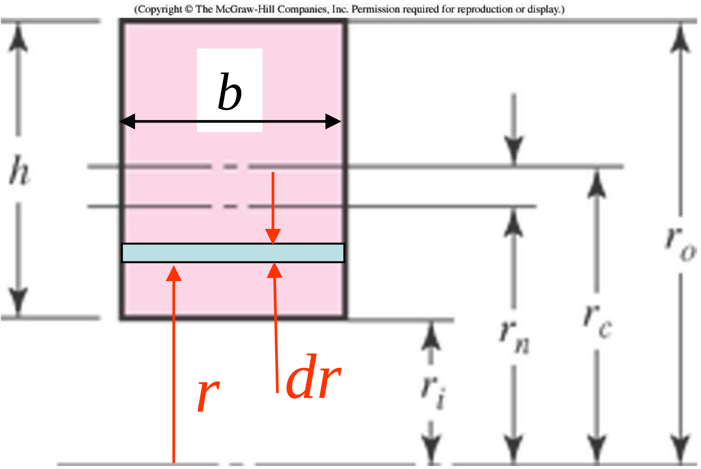
If the bending moment M is due to a force F, the moment is taken about centroidal axis rather than neutral axis. Also, the direct normal stress $\pm F/A$ should be superposed with bending stress.



In curved beam problems, one needs to take the integrals;

$$r_n = \frac{A}{\int_A \frac{dA}{r}}$$ and $$r_c = \frac{\int_A r dA}{A}$$

For some other common cross sections (Circular, trapezoidal, I , box) the integrals are listed in a table, in the text book.)



$$r_n = \frac{A}{\int_A \frac{dA}{r}} = \frac{bh}{\int_A \frac{bdr}{r}} = \frac{h}{\ln \frac{r_o}{r_i}}$$

$$r_c = r_i + h/2$$

Optimum cross section:

The cross section is most efficiently utilized when the maximum tensile and compressive stresses are equal.

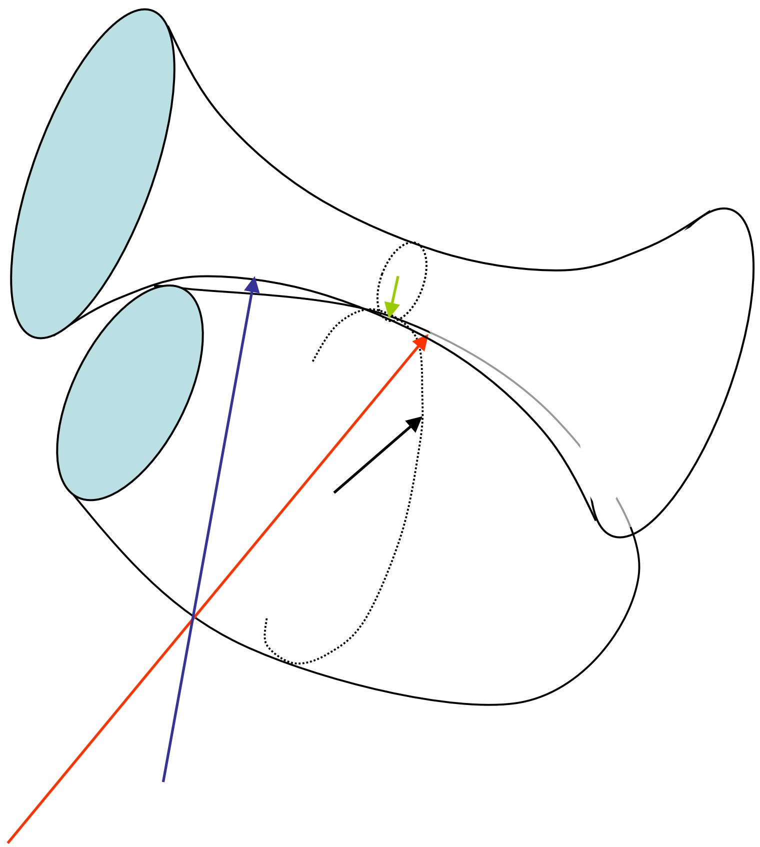
Then for the case of pure bending,

$$\sigma_i = \sigma_o$$ and $$\frac{c_i}{c_o} = \frac{r_i}{r_o}$$

Contact Stresses

- When two bodies having curved surfaces are pressed together;
  - point or line contact changes to area contact
  - 3D stress state develops in the vicinity of contact area (local stresses)
  - stresses may cause failure such as cracking, pitting, flaking.
- Examples: wheel on rail, cam-follower, pin in a bearing, mating gear teeth, rollers on raceway of an anti-friction bearing.

Most general case occurs when both bodies have double radius of curvature.



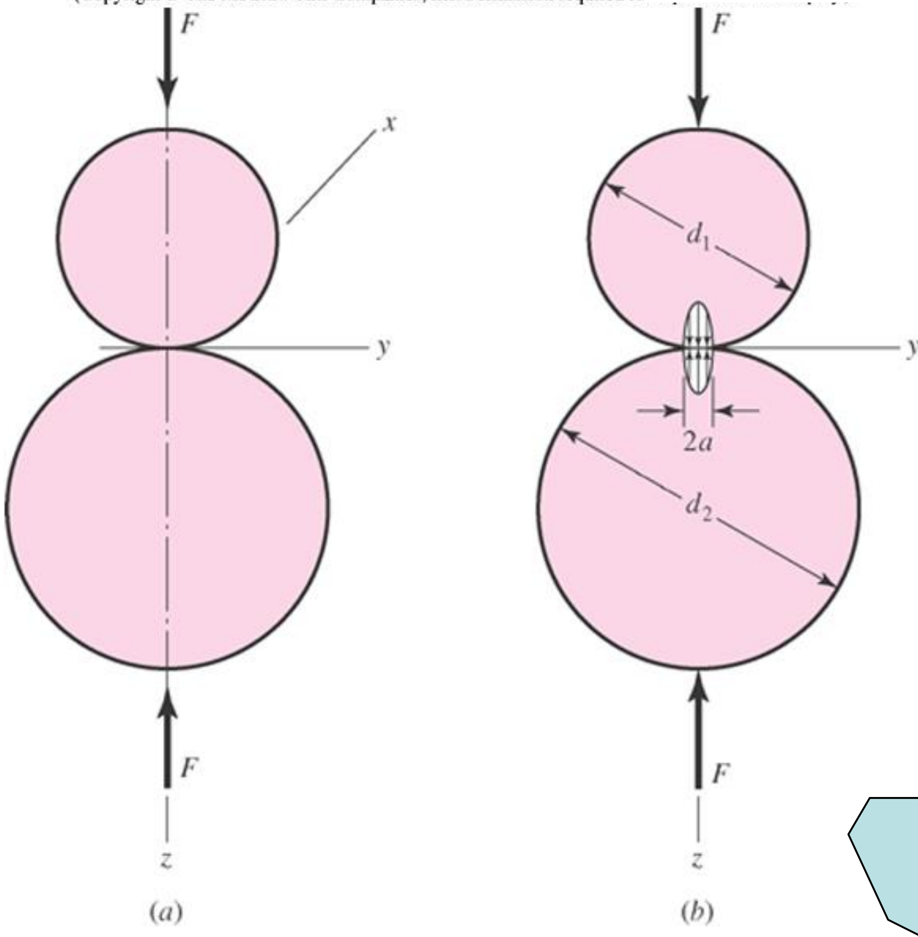
We consider only two special cases of practical importance:

i) Contacting spheres

ii) contacting cylinders

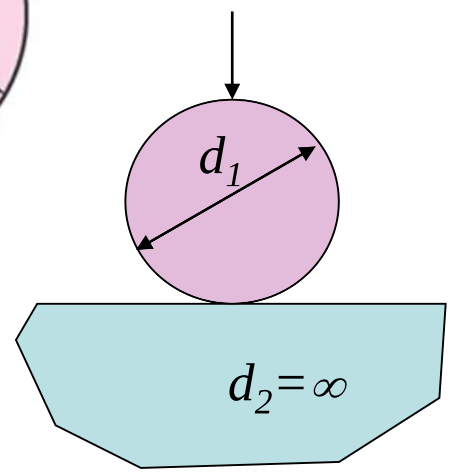
The results presented here are due to Hertz, so the contact stresses are also called Hertzian stresses.

<u>Contacting Spheres:</u> Hemispherical pressure distribution is formed. Max. pressure is at the center of the circular contact area, of diameter 2*a*.

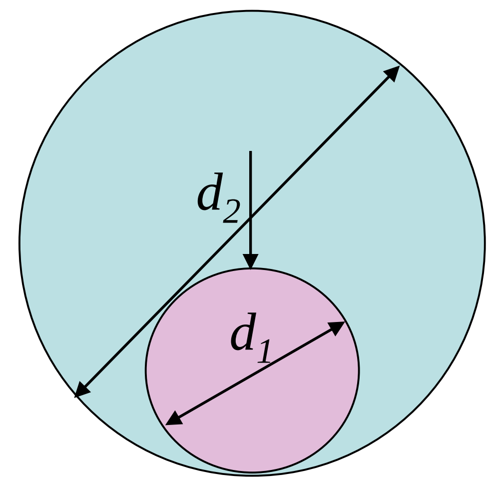


$$a = \sqrt[3]{\frac{3F}{8}\,\frac{(1-\nu_1^{\,2})/E_1 + (1-\nu_2^{\,2})/E_2}{1/d_1 + 1/d_2}}$$

$$P_{\max} = \frac{3F}{2\pi a^2}$$



$1/d_2 = 0$



$d_2 < 0$

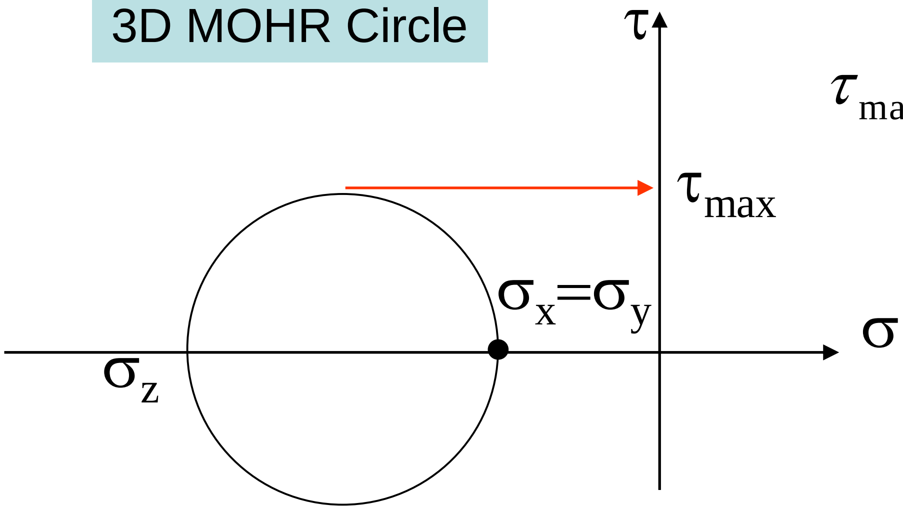
Maximum stresses occur on z-axis and these are principal stresses.

$$\sigma_x = \sigma_y = -p_{\max}\left[(1+\nu)\left(1-\left|\frac{z}{a}\right|\tan^{-1}\left(\frac{1}{|z/a|}\right)\right)-\frac{1}{2(1+z^2/a^2)}\right]$$

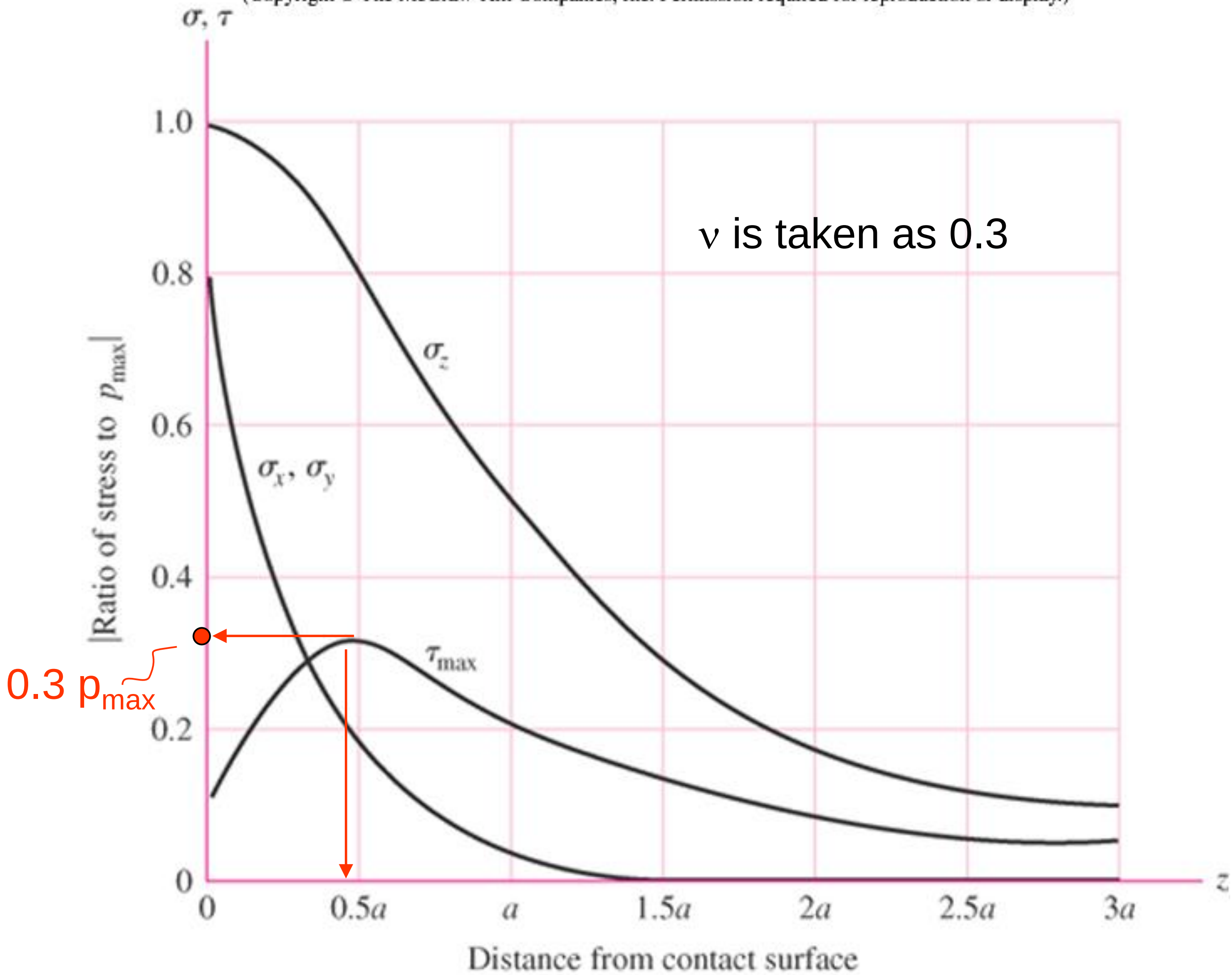
$$\sigma_z = -\frac{p_{\max}}{1+z^2/a^2}$$

we considered stresses only on z-axis (x=y=0) for design and analysis purposes, since the maximum stresses and the critical point (the point which is most likely to fail) are on z-axis.

3D MOHR Circle

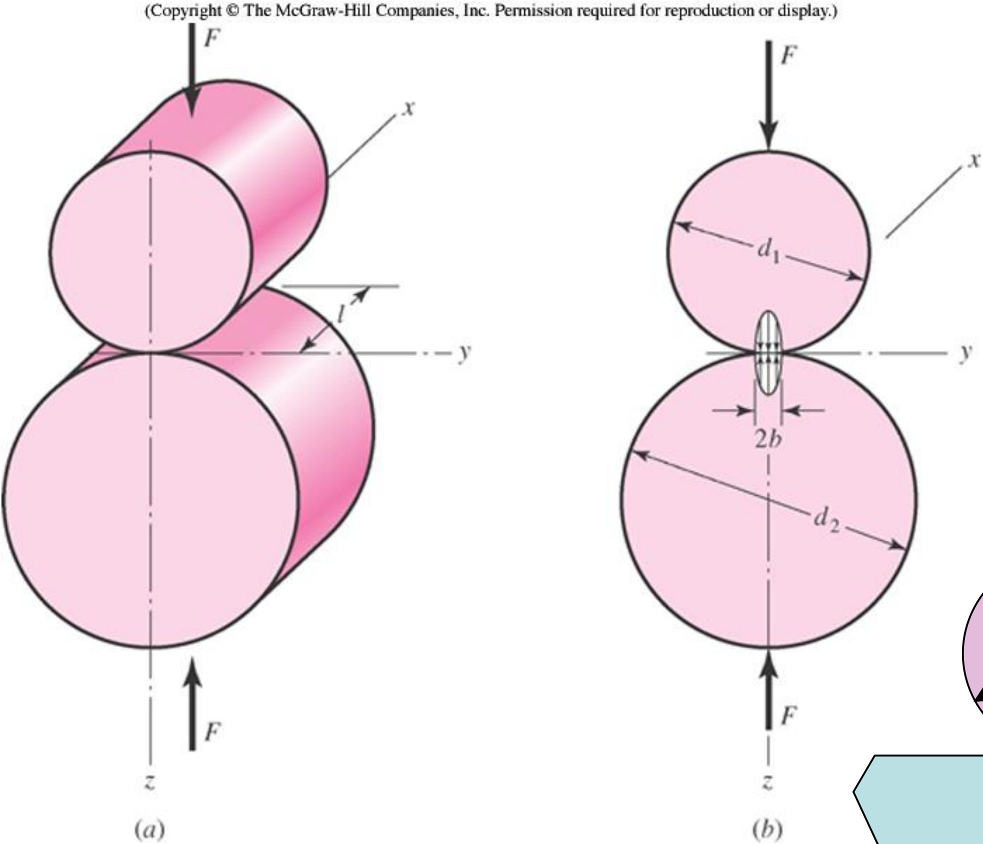


$$\tau_{\max} = \frac{\sigma_x - \sigma_z}{2} = \frac{\sigma_y - \sigma_z}{2}$$



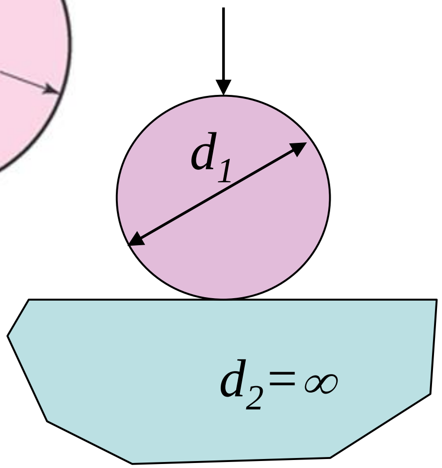
The crack originates at the location of maximum shear stress, which is slightly below the surface, and propagates towards the surface.

Contacting Cylinders: Elliptical pressure distribution is formed. Max. pressure is at y=0.

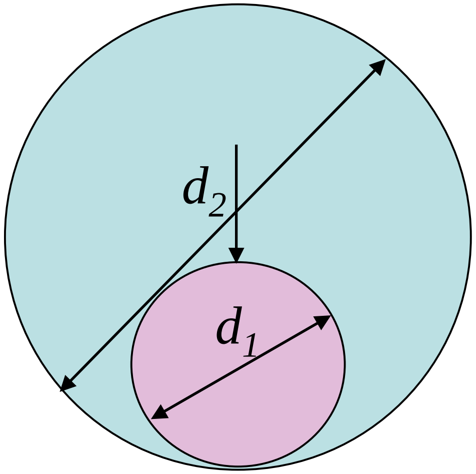


$$b = \sqrt{\frac{2F}{\pi\ell}\frac{(1-\nu_1^{\,2})/E_1 + (1-\nu_2^{\,2})/E_2}{1/d_1 + 1/d_2}}$$

$$P_{\max} = \frac{2F}{\pi b \ell}$$



$1/d_2=0$



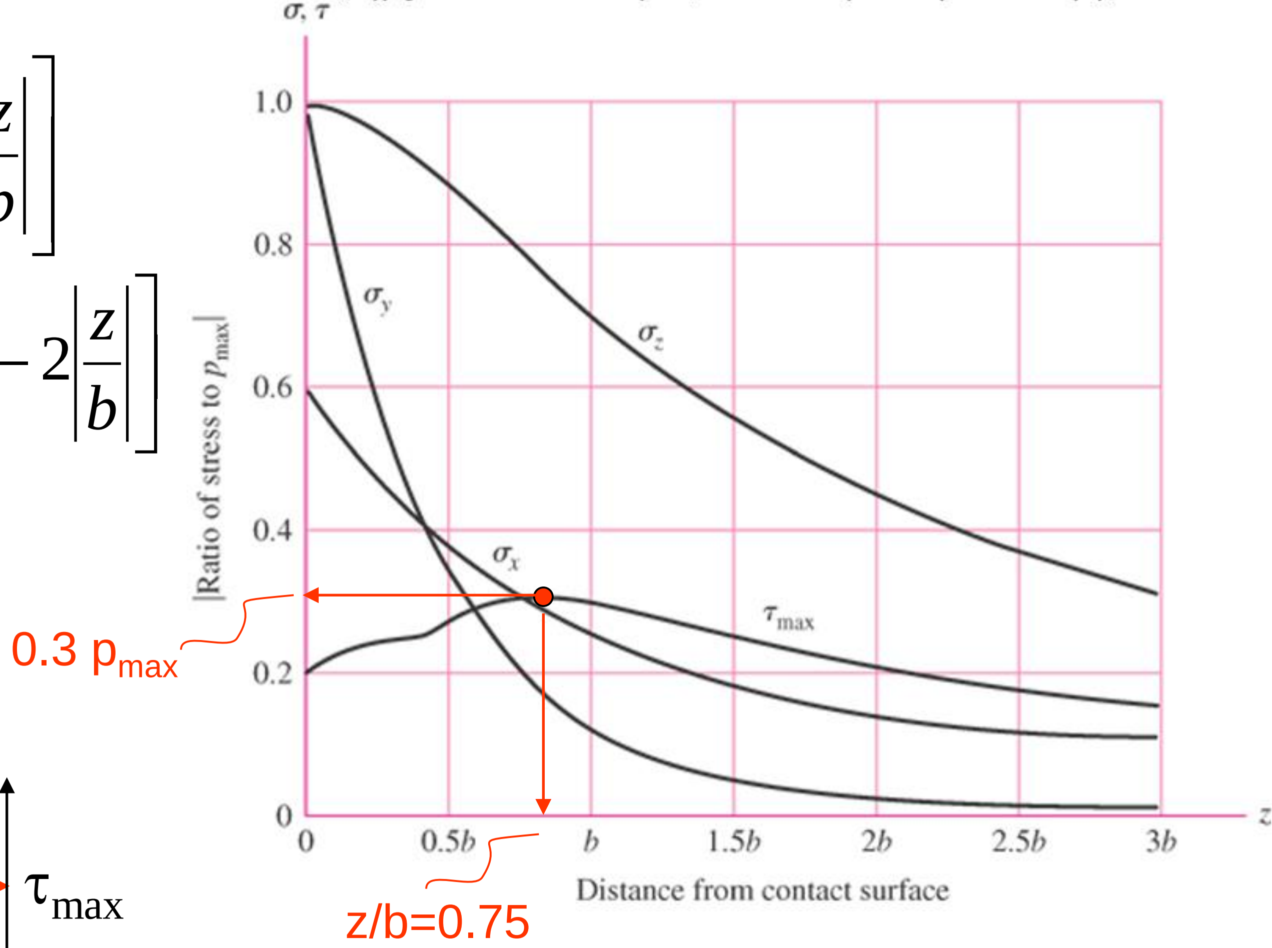
$d_2<0$

Stresses along z-axis :

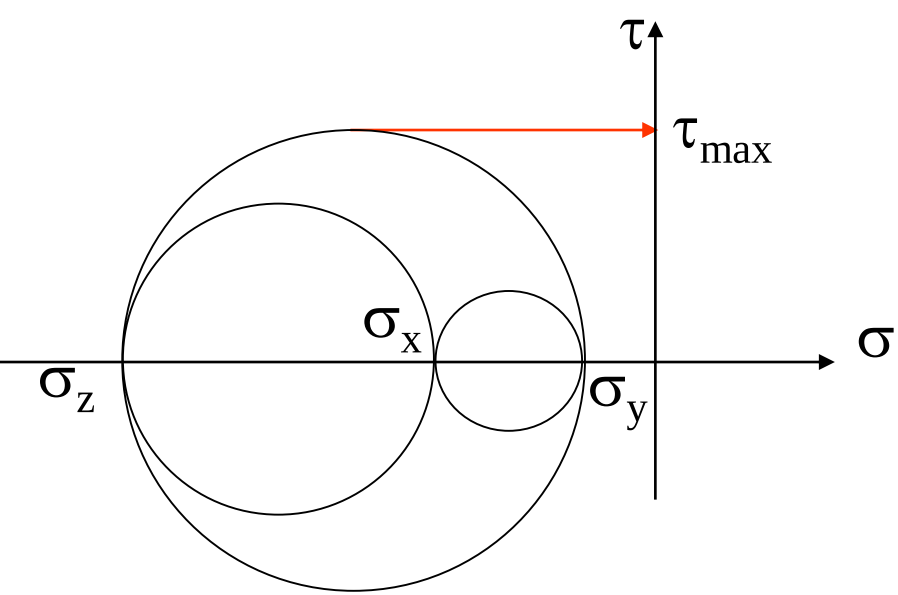
$$\sigma_x = -2\nu p_{\max}\left[\sqrt{1+\frac{z^2}{b^2}} - \left|\frac{z}{b}\right|\right]$$

$$\sigma_y = -p_{\max}\left[\frac{1+2\left(z^2/b^2\right)}{\sqrt{1+z^2/b^2}} - 2\left|\frac{z}{b}\right|\right]$$

$$\sigma_z = -\frac{p_{\max}}{\sqrt{1+z^2/b^2}}$$



3D MOHR Circle



$$\tau_{\max} = \frac{\sigma_z - \sigma_y}{2}$$