Stress Analysis
Part I

STRESS ANALYSIS

Design of many machine elements is governed by the stress state at the critical points.

Hence the designer should be able to:

- Determine the critical points
- Determine the stress state at these points

Critical point(s) : The points on the machine element which are most likely to fail under given loads.

Failure : Yielding or Fracture (In this context)

Question : How do we identify critical points then?

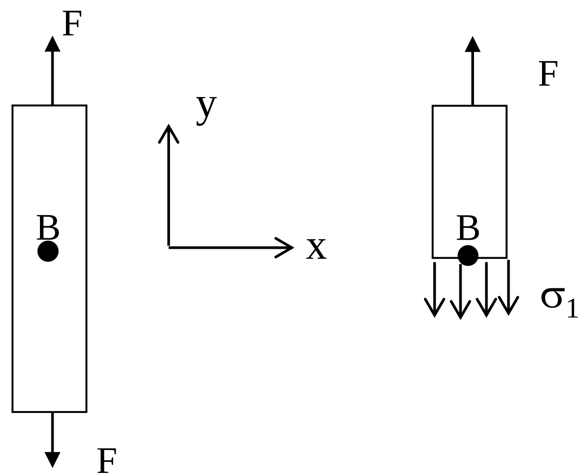
Answer : Locations of maximum stress, locations of minimum strength, and locations where stresses are high and strength is low are candidates.

Example: For an end loaded cantilever beam, critical points are at the location of support at the top and bottom surfaces of the beam, since the bending moment and consequently the bending stress are maximum there.

Furthermore if the beam has different tensile and compressive strengths (like concrete) one of these locations may be even more critical.

NOTE THAT strength as well as stresses can change from point to point on a body. Hence there may be more than one critical point on the body. All of them should be checked.

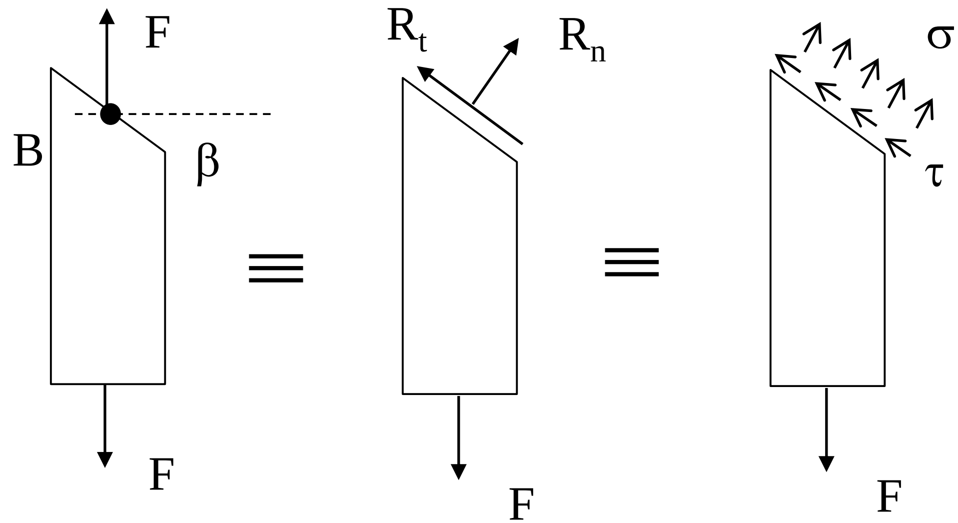
Stress State:
Consider a point B on a prismatic member. Take a cut perpendicular to the axis of the member through point B.



Resultant of normal stress over the cross sectional area, Ac, gives force F.

$\sigma_1 = F/A_c$

Now take a cut through B, which makes an angle β with the x-axis.

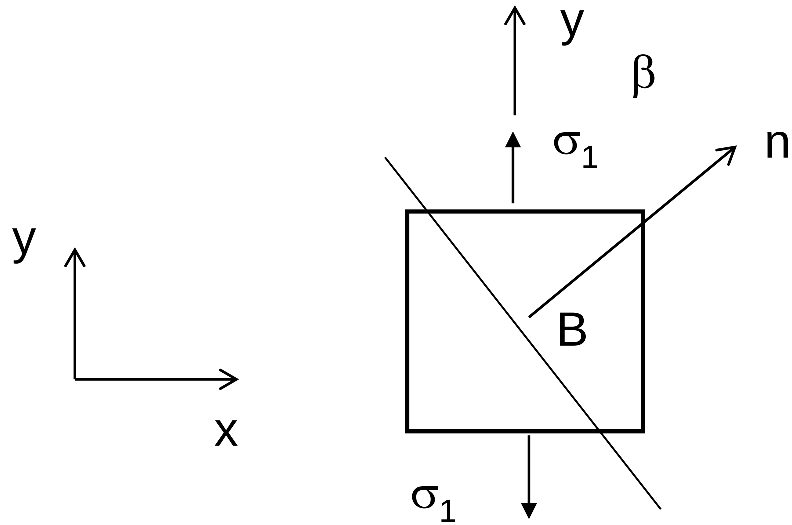


F acting on A'_c is produced as the resultant of R_t and R_n. In turn R_t and R_n are the resultants of stresses σ and τ acting on A'_c.

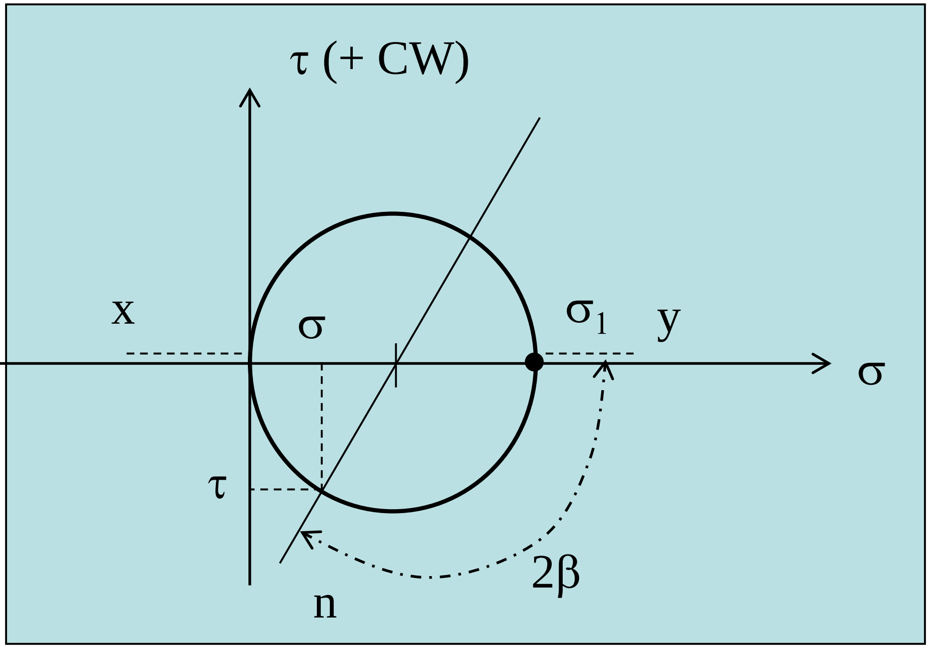
Question: How are σ and τ at point B related to σ_1 ?

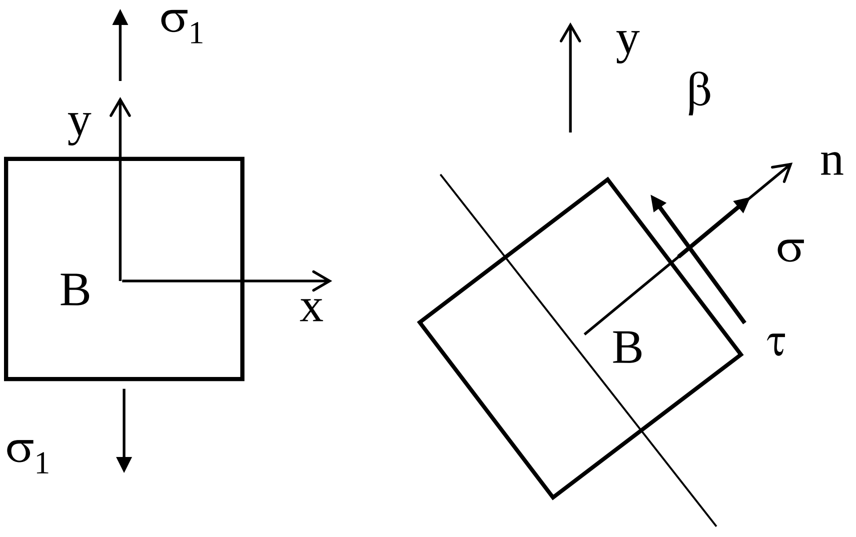
Answer: Using static equilibrium equations or MOHR CIRCLE

Consider a very small cubic element around point B.

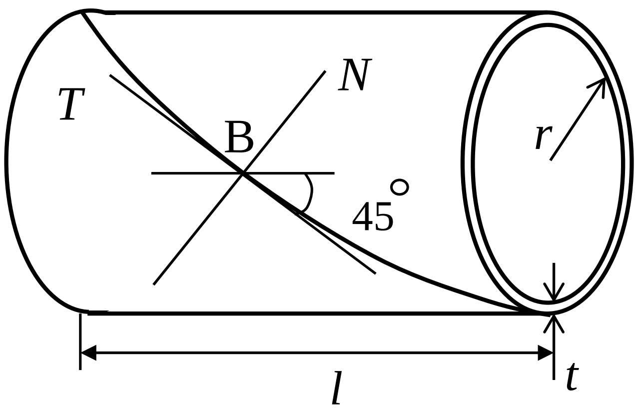


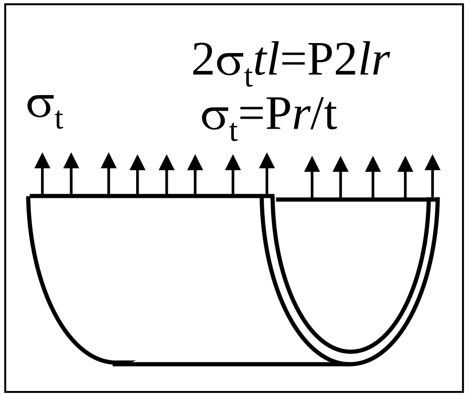
Given the uniaxial stress state σ_1. We want to find the normal and shear stresses on the surface whose normal is **n**.

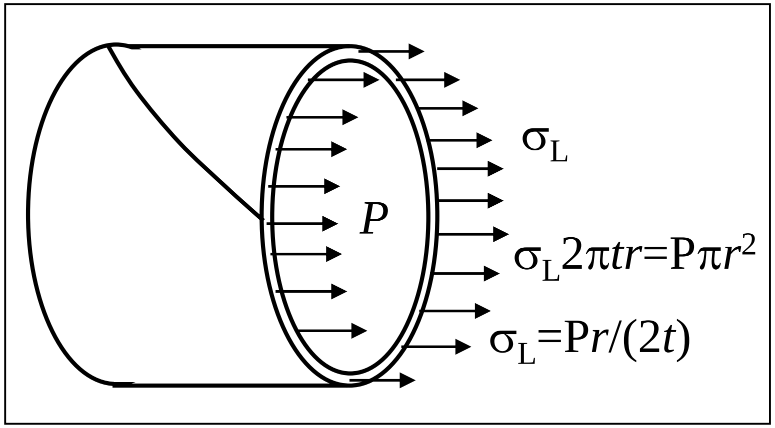




Example: Consider a thin walled cylindrical pressure vessel with a helical weld line. Let it be given that the weld line is critical. σ_{all} (on weld line), τ_{all} (on weld line), r and t are given. Maximum safe internal pressure is asked.







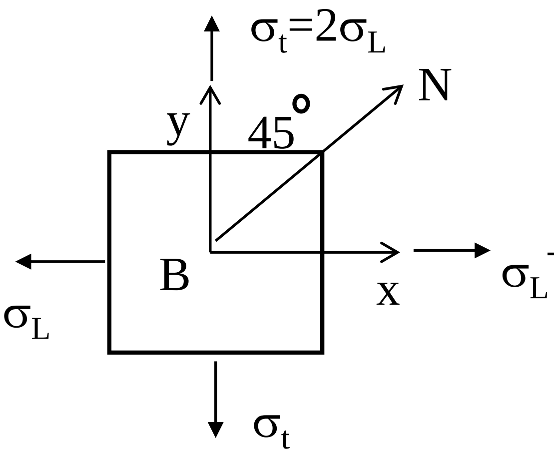
Bi-axial stress state

Mohr Circle

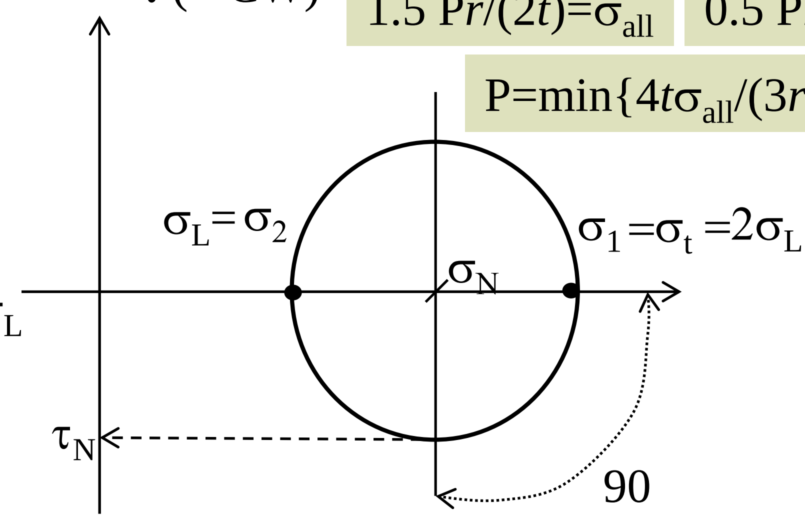
$\sigma_n = (\sigma_L + 2\sigma_L)/2 = 1.5\sigma_L$ $\tau_n = (2\sigma_L - \sigma_L)/2 = 0.5\sigma_L$

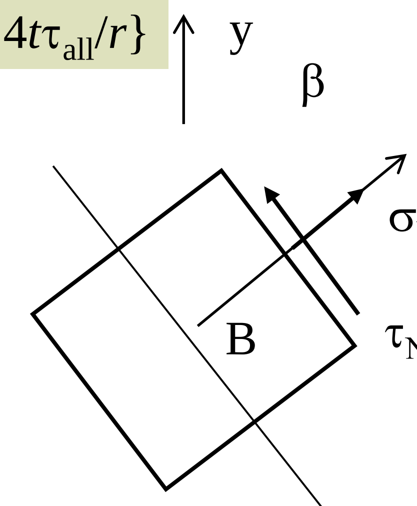
$1.5\, Pr/(2t) = \sigma_{all}$ $0.5\, Pr/(2t) = \tau_{all}$

$P = \min\{4t\sigma_{all}/(3r),\ 4t\tau_{all}/r\}$



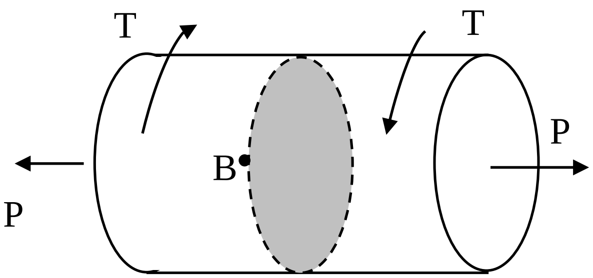
τ (+ CW)

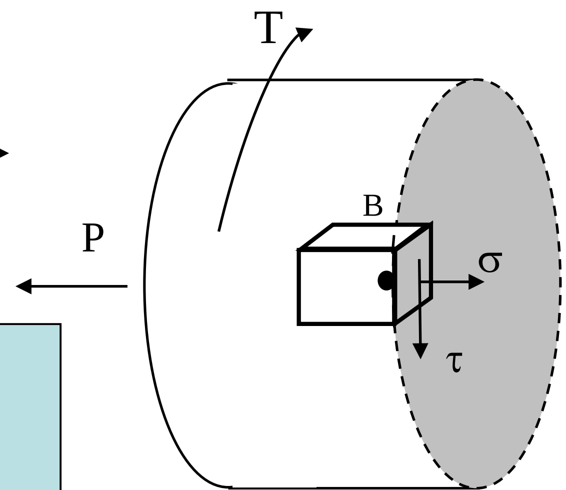


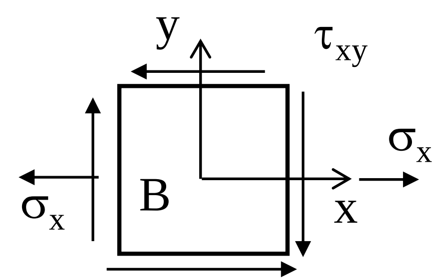


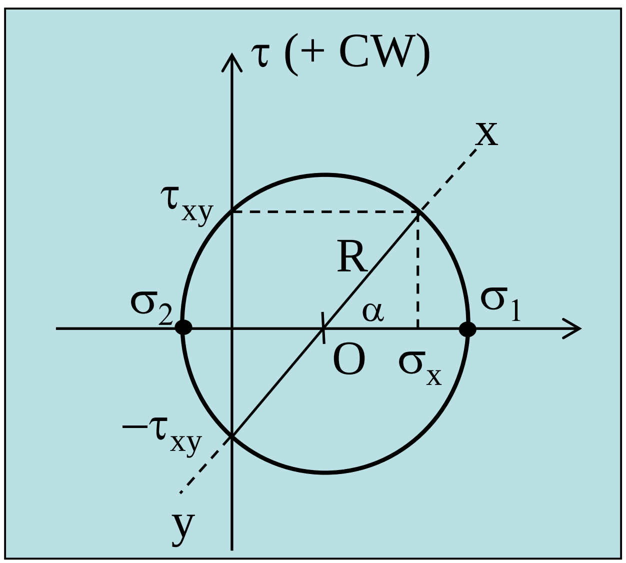
<u>Example:</u>Consider a cylinder subjected to torsion and tension

Points on the surface are critical. $\tau = Tr/J$ $\sigma = P/A$ Take a section through B.







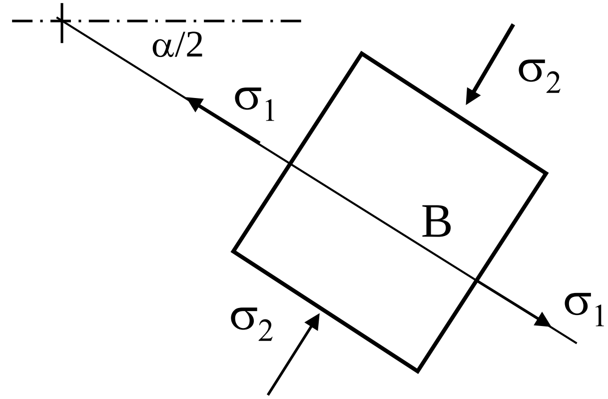


$O{:}(\sigma_x/2, 0)$

$R = [(\sigma_x/2)^2 + \tau_{xy}{}^2]^{1/2}$

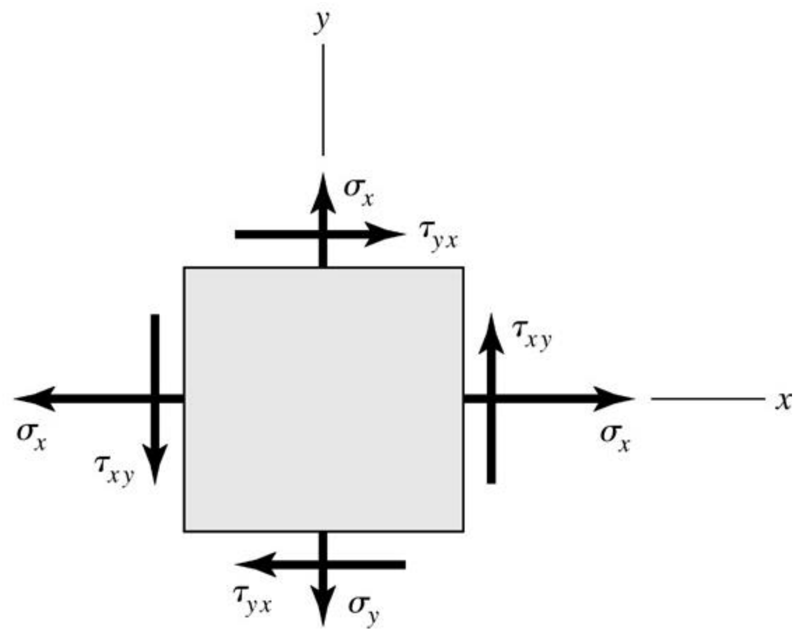
$\alpha = \tan^{-1}(2\tau_{xy}/\sigma_x)$

$\sigma_{1,2} = (\sigma_x/2) \pm [(\sigma_x/2)^2 + \tau_{xy}{}^2]^{1/2}$



Upto this point all the cases considered were two dimensional. (Bi-axial)
The stresses considered were all acting on planes whose normals are in x-y planes. (stresses are in x-y plane)

Hence stresses can be represented on a cubic element as in the figure or in an array form as shown below.

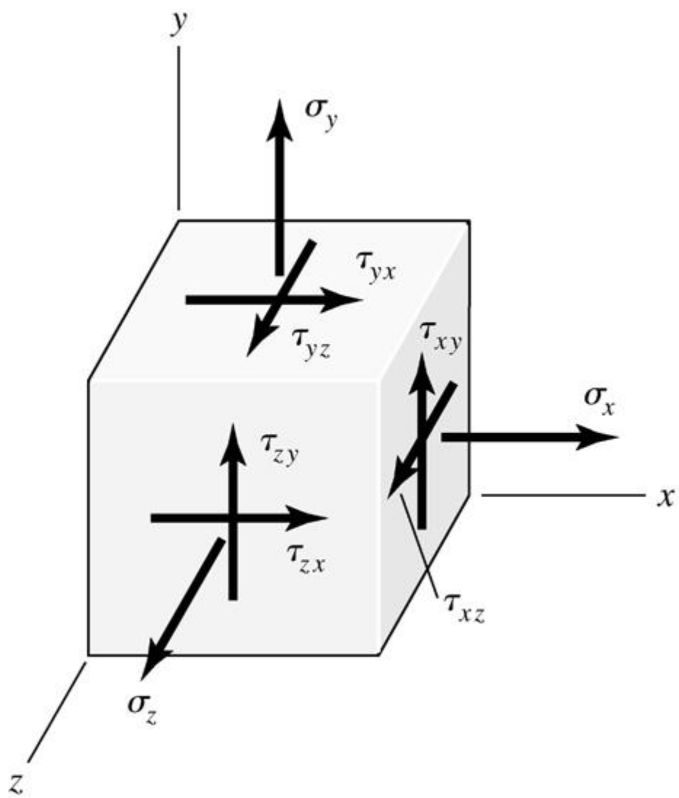


$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{bmatrix}$$

The cube and/or the array represent the 2-D stress state at a point.

By using equilibrium it can be shown that $\tau_{xy}=\tau_{yx}$.

In the general 3-D case, there will be 3 more stress components as shown in the figure.



3-D stress state at a point is also represented by the array below.

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

Note that $\tau_{xy}=\tau_{yx}$, $\tau_{xz}=\tau_{zx}$, $\tau_{zy}=\tau_{yz}$.

<u>Name convention:</u> σ_{ij}

i: direction of normal vector of the surface on which stress acts.

j:direction of stress itself

<u>Sign convention:</u>

- tensile stresses are (+),
- compressive stresses are (–).

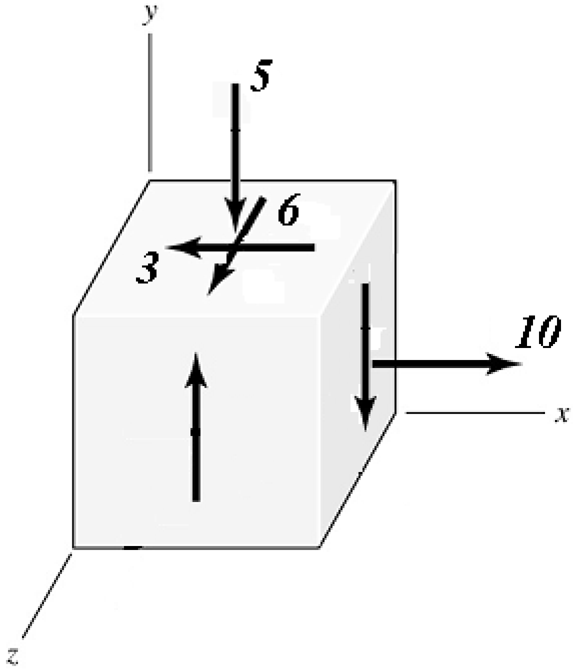
A shear stress is (+) if it is in;

1. (+) direction, on a surface whose normal is in (+) direction.
2. (–) direction, on a surface whose normal is in (–) direction.

Otherwise shear stress is (–).

All the stresses shown above are positive.

Example:



$\sigma_{xx}=10 \quad \sigma_{zz}=0$

$\sigma_{yy}=-5 \quad \tau_{yz}=6$

$\tau_{xy}=-3 \quad \tau_{xz}=0$

$$\begin{bmatrix} 10 & -3 & 0 \\ -3 & -5 & 6 \\ 0 & 6 & 0 \end{bmatrix}$$

Principal stresses in 3-D are obtained by solving the following eigen value problem:

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} \ell \\ m \\ n \end{Bmatrix} = \sigma \begin{Bmatrix} \ell \\ m \\ n \end{Bmatrix}$$

where

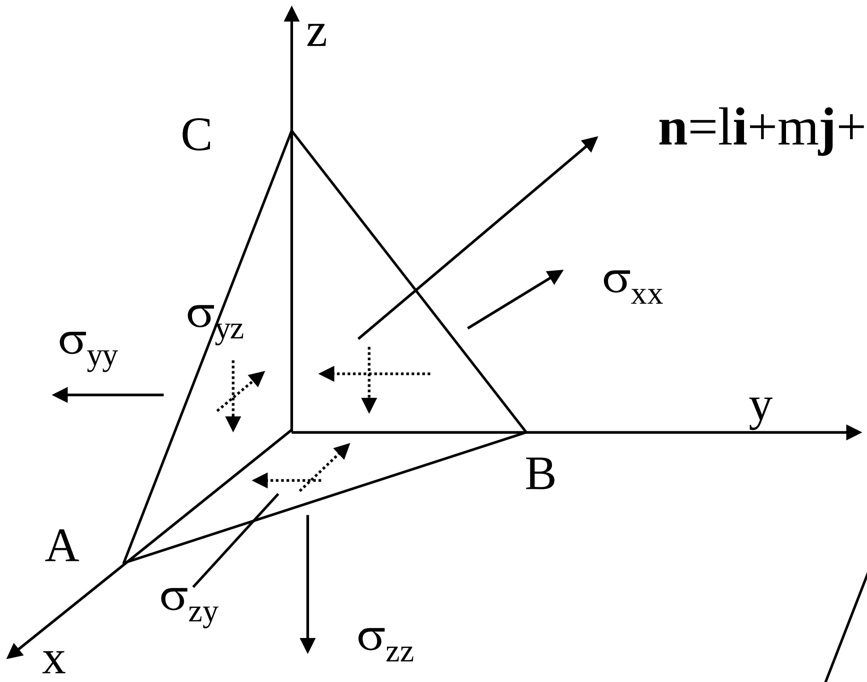
$\begin{Bmatrix} \ell \\ m \\ n \end{Bmatrix}$ is the direction cosine vector, (eigen vector) of principal stress, σ (eigen value). Therefore,
$$\begin{vmatrix} \sigma_{xx}-\sigma & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy}-\sigma & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz}-\sigma \end{vmatrix} = 0$$

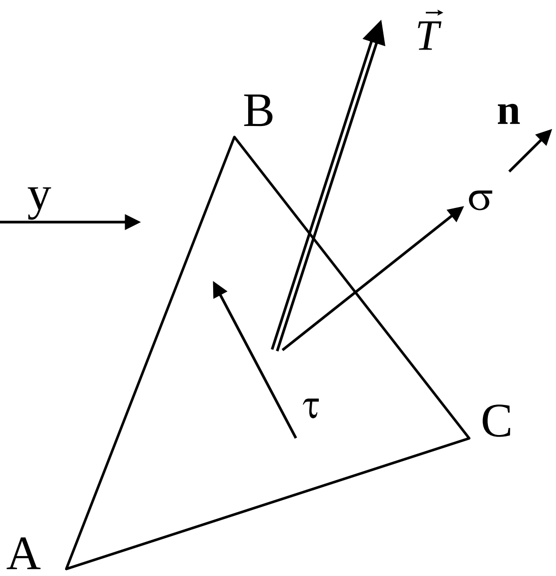
then characteristic equation is

$$\sigma^3 - (\sigma_{xx}+\sigma_{yy}+\sigma_{zz})\sigma^2 + (\sigma_{xx}\sigma_{yy}+\sigma_{xx}\sigma_{zz}+\sigma_{yy}\sigma_{zz}-\sigma_{yx}{}^2-\sigma_{yz}{}^2-\sigma_{xz}{}^2)\sigma - \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{vmatrix} = 0$$

A simple special case of 3D stress state occurs when two opposite sides of the stress element are free from shear stresses. Then, the normal stress on these faces is a principal stress. The two other principle stresses can be found by 2D Mohr Circle analysis. (This is because superposition is applicable.)

Stresses at a point acting on a given plane is found from equilibrium of an infinitessimal element, shown below. $\Sigma F_x=0,\ \Sigma F_y=0, \Sigma F_z=0.$





$$\vec{T} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} \ell \\ m \\ n \end{Bmatrix}$$

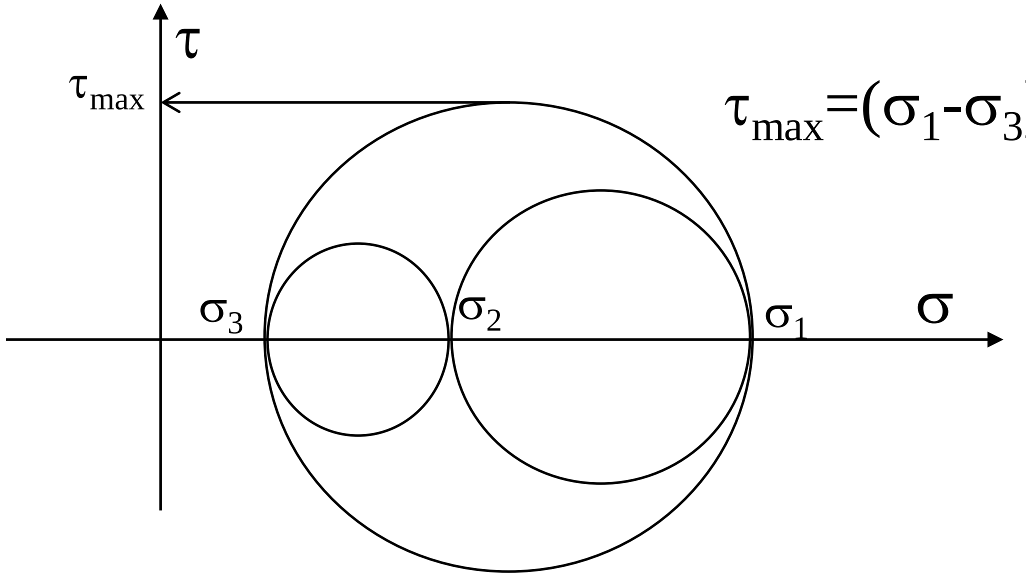
, $\vec{T}$ is the traction vector.

$$\sigma = \vec{T}.\vec{n} \quad , \quad \tau = \sqrt{T^2 - \sigma^2}$$

where $T = \left|\vec{T}\right|$.

In 3-D (triaxial stress state) Mohr circle can be drawn only after finding the principal stresses by solving the eigen value problem.

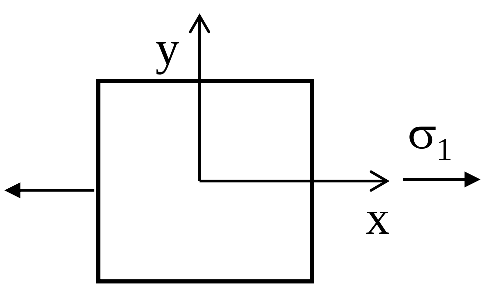
let $\sigma_1>\sigma_2>\sigma_3$ be the principal stresses. Max. shear stress is given by the largest of the 3 circles.

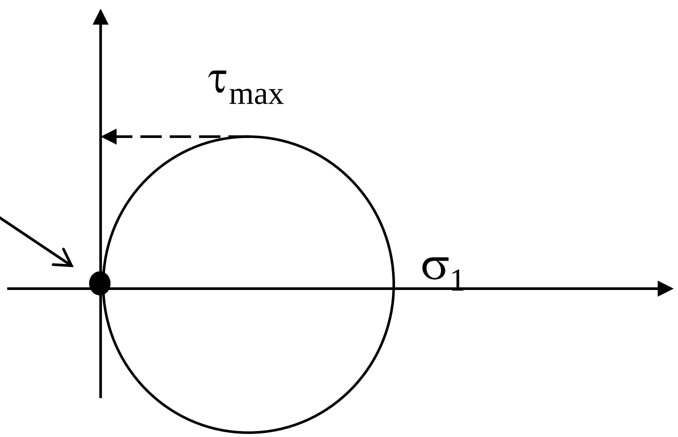


$\tau_{max}=(\sigma_1-\sigma_3)/2$

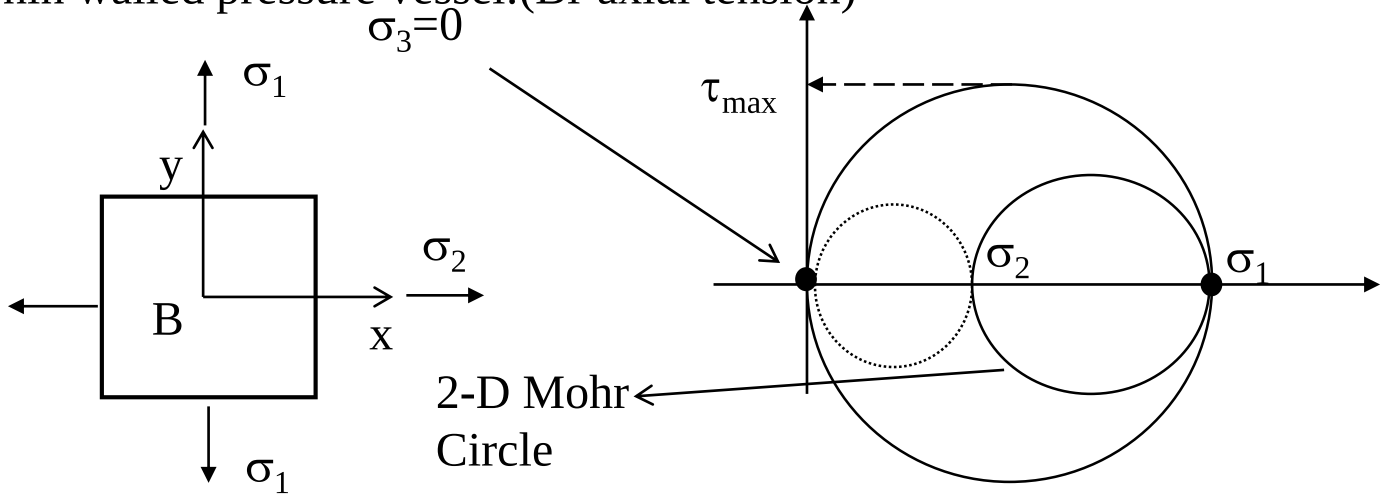
2D Cases which we have considered earlier are special cases of the general 3-D case where one (or two) of the principal stresses is zero.

Uniaxial tension: $\sigma_2=\sigma_3=0$

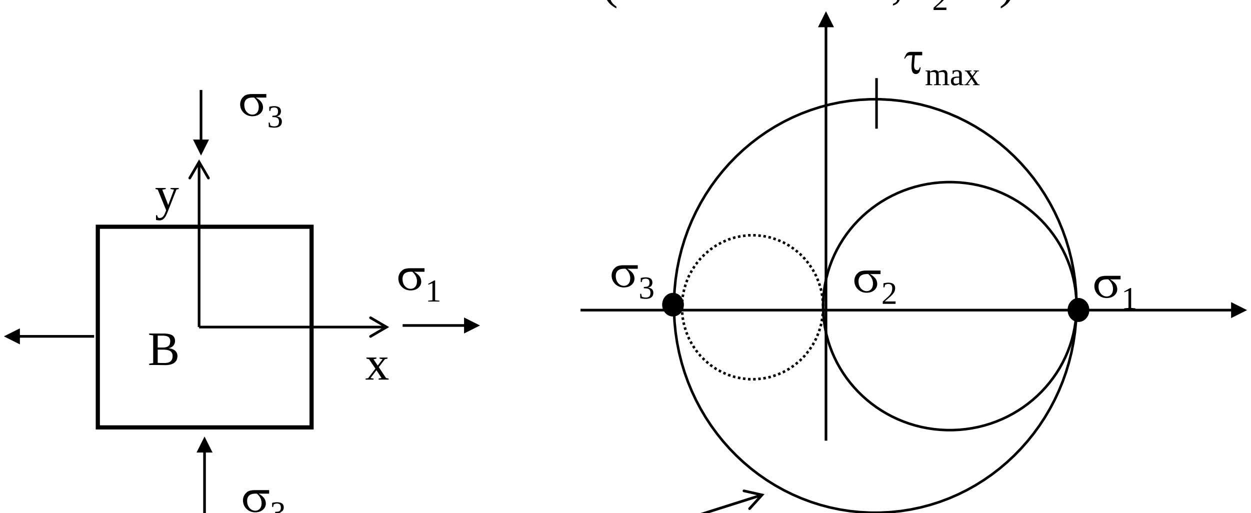




Thin walled pressure vessel:(Bi-axial tension)

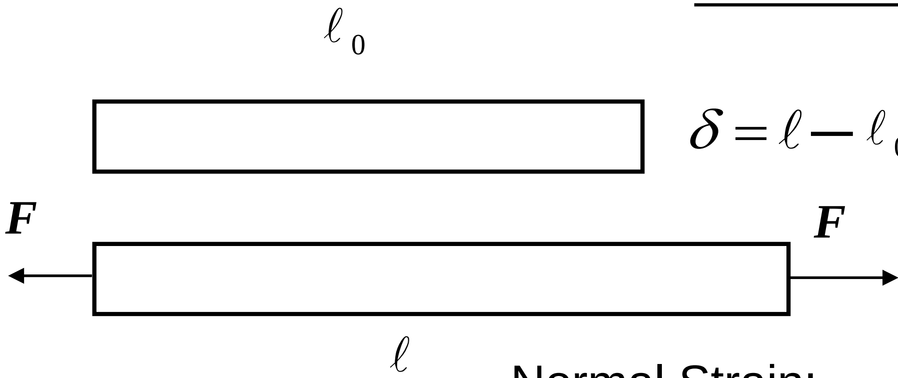


Combined tension and torsion:(Bi-axial stress,σ_2=0)

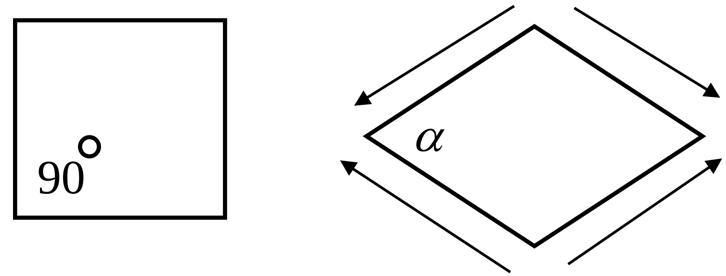


Note that the 2-D Mohr circle drawn may or may not give the maximum shear stress (which is used in design criteria). Therefore always draw the 3-D Mohr circle by taking one principal stress as zero after a two dimensional analysis to find τ_{max}.

STRAIN



$$\delta = \ell - \ell_0$$



Normal Strain:

$$\varepsilon = \frac{\ell - \ell_0}{\ell_0} = \frac{\delta}{\ell_0}$$

Shear Strain:

$$\gamma = \frac{\pi}{2} - \alpha$$

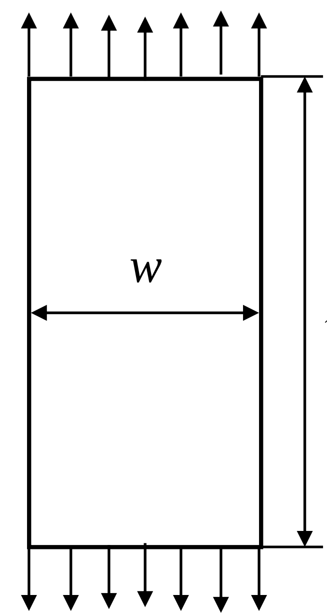
HOOKE'S LAW:

Uniaxial loading: $\sigma = E\varepsilon$ *E* is Young's Modulus.

Shear loading: $\tau = G\gamma$ *G* is Modulus of Rigidity.

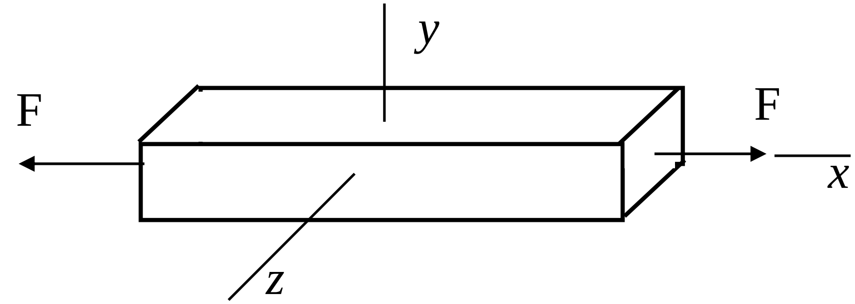
$$\nu = -\frac{\text{lateral strain}}{\text{axial strain}}$$ (loading is axial)

ν is called Poisson Ratio. $\nu = -\dfrac{\Delta w / w}{\Delta \ell / \ell}$ $-1 < \nu < 0.5$

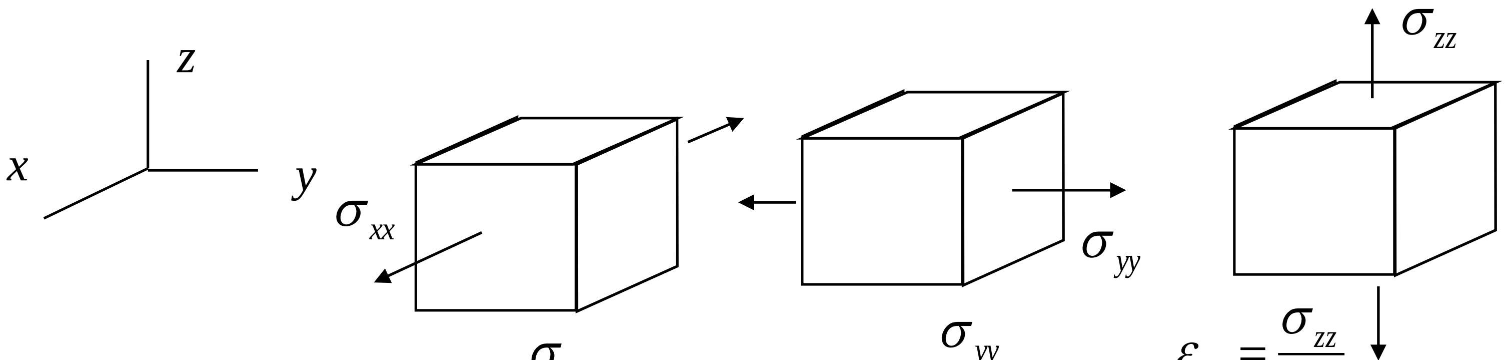


0.5 corresponds to incompressible materials.

$$\varepsilon_{yy} = -\nu\varepsilon_{xx} \qquad \varepsilon_{zz} = -\nu\varepsilon_{xx} \qquad G = \frac{E}{2(1+\nu)}$$



General Stress Strain relations for İsotropic Homogeneous Materials:



$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} \qquad \varepsilon_{yy} = \frac{\sigma_{yy}}{E} \qquad \varepsilon_{zz} = \frac{\sigma_{zz}}{E}$$

$$\varepsilon_{xx} = -\nu\varepsilon_{yy} \qquad \varepsilon_{xx} = -\nu\varepsilon_{zz}$$

By superposition:

$$\varepsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu\sigma_{yy} - \nu\sigma_{zz}\right) \qquad \varepsilon_{xx} = -\nu\frac{\sigma_{yy}}{E} \qquad \varepsilon_{xx} = -\nu\frac{\sigma_{zz}}{E}$$

$$\varepsilon_{yy} = \frac{1}{E}\left(\sigma_{yy} - \nu\sigma_{xx} - \nu\sigma_{zz}\right) \qquad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

$$\varepsilon_{zz} = \frac{1}{E}\left(\sigma_{zz} - \nu\sigma_{yy} - \nu\sigma_{xx}\right) \qquad \gamma_{xz} = \frac{\tau_{xz}}{G} \qquad \gamma_{yz} = \frac{\tau_{yz}}{G}$$

Mohr Circle for strains can be drawn similar to stresses but τ must be replaced with $\gamma/2$, σ must be replaced with ε.

PLANE STRESS and PLANE STRAIN problems

Plane Stress: $\sigma_{zz}=\tau_{xz}=\tau_{yz}=0$

Thickness of the member in z-direction is small, loading is in x-y plane.

$$\varepsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu\sigma_{yy}\right) \qquad \varepsilon_{yy} = \frac{1}{E}\left(\sigma_{yy} - \nu\sigma_{xx}\right)$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \qquad \varepsilon_{zz} = -\frac{\nu}{E}(\sigma_{xx} + \sigma_{yy})$$

One can solve σ_{xx} and σ_{yy} in terms of ε_{xx} and ε_{yy} to obtain,

$$\sigma_{xx} = \frac{E}{1-\nu^2}\left(\varepsilon_{xx} + \nu\varepsilon_{yy}\right)$$

$$\sigma_{yy} = \frac{E}{1-\nu^2}\left(\varepsilon_{yy} + \nu\varepsilon_{xx}\right)$$

Plane Strain: $\varepsilon_{zz}=0$ (or constant in generalized p. strain) , $\gamma_{xz}=\gamma_{yz}=0$

Thickness of the member in z-direction is very large, loading is in x-y plane and it is not a function of z.

$$\varepsilon_{zz} = 0 = \frac{1}{E}\left(\sigma_{zz} - \nu\sigma_{yy} - \nu\sigma_{xx}\right)$$

$$\sigma_{zz} = \nu(\sigma_{yy} + \sigma_{xx})$$

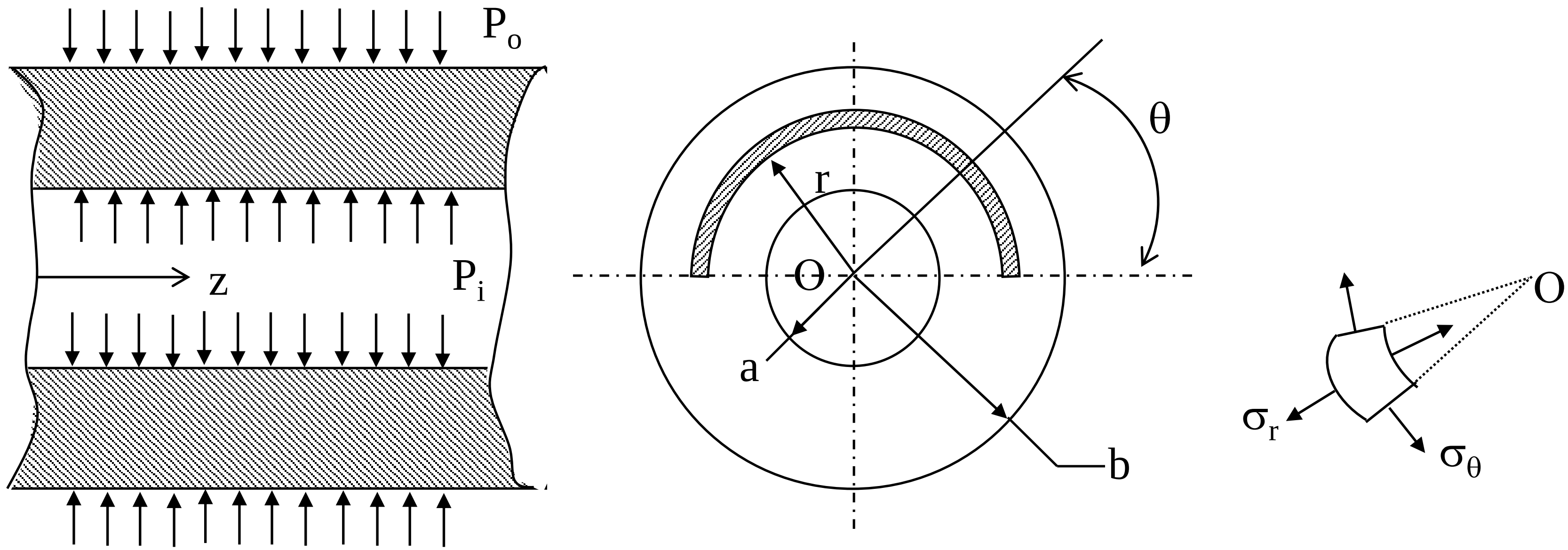
$$\varepsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu\sigma_{yy} - \nu^2(\sigma_{xx} + \sigma_{yy})\right)$$

$$\varepsilon_{xx} = \frac{1-\nu^2}{E}\left(\sigma_{xx} - \frac{\nu}{1-\nu}\sigma_{yy}\right)$$

$$\varepsilon_{yy} = \frac{1-\nu^2}{E}\left(\sigma_{yy} - \frac{\nu}{1-\nu}\sigma_{xx}\right) \qquad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

One can again solve σ_{xx} and σ_{yy} in terms of ε_{xx} and ε_{yy}.

THICK WALLED CYLINDERS



Stress state at a point in the cylinder is to be determined.

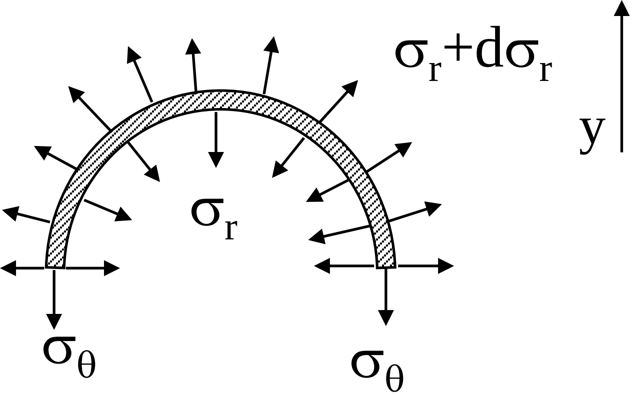
Assumptions:

1) ε_{zz} is constant. (Plane sections remain plane subsequent to loading.)

2) Problem is axisymmetric. (Stresses and diplacements are independent of θ. $\sigma_{r\theta}=0$)

3) $\sigma_{rz}=\sigma_{z\theta}=0$

4) Ends are free. ($\sigma_{zz}=0$, If ends were close, σ_{zz} would be constant.)



$\Sigma Fy=0: (\sigma_r+d\sigma_r)\ 2\ (r+dr)-2\ r\ \sigma_r-2\ dr\ \sigma_\theta=0$

$$\boxed{r d\sigma_r/dr+\sigma_r-\sigma_\theta=0 \quad .....(1)}$$

(Note dr^2 terms are neglected)

Consider ε_{zz}: $\quad \varepsilon_{zz}=c'=\frac{1}{E}\left(\cancelto{0}{\sigma_z}-\nu\sigma_r-\nu\sigma_\theta\right) \longrightarrow \sigma_r+\sigma_\theta=C\ (=Ec'/\nu)$

then $\boxed{\sigma_\theta=C-\sigma_r \ .....(2)}$

From (1) and (2),

$r d\sigma_r/dr+2\sigma_r=C$ (1st order O.D.E.) Solution is given as follows:

$\sigma_r=C/2+B/r^2$ and from (2), $\sigma_\theta=C/2-B/r^2$ Let $C/2=A$, and we obtain;

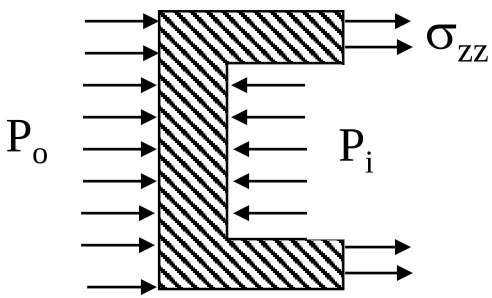
$$\boxed{\begin{aligned}\sigma_r&=A+B/r^2\\ \sigma_\theta&=A-B/r^2\end{aligned}}$$

Boundary Conditions:

$\sigma_r = -P_i$, $r=a$, $\sigma_r = -P_o$, $r=b$ then, $A = \dfrac{P_i a^2 - P_o b^2}{b^2 - a^2}$ $B = \dfrac{a^2 b^2 (P_o - P_i)}{b^2 - a^2}$

$$\boxed{\sigma_r = \frac{P_i a^2 - P_o b^2}{b^2 - a^2} + \frac{1}{r^2}\frac{a^2 b^2 (P_o - P_i)}{b^2 - a^2} \qquad \sigma_\theta = \frac{P_i a^2 - P_o b^2}{b^2 - a^2} - \frac{1}{r^2}\frac{a^2 b^2 (P_o - P_i)}{b^2 - a^2}}$$

If ends of the cylinder are closed

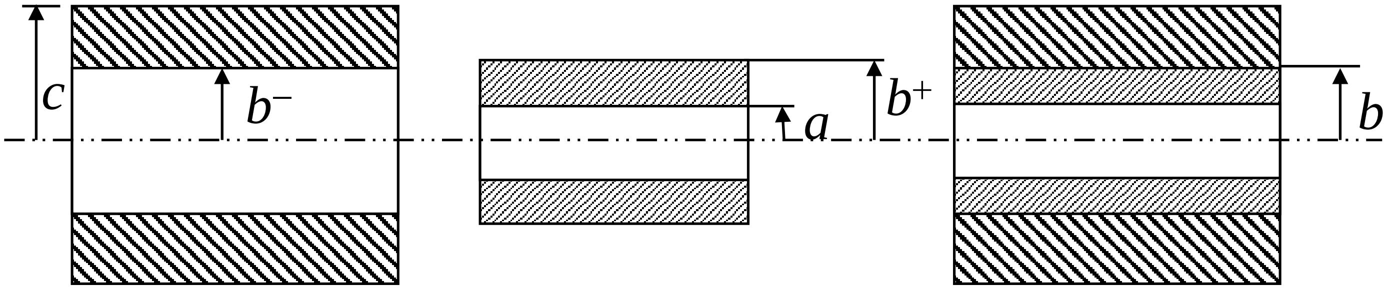


$$-\sigma_{zz}(b^2-a^2) - P_o b^2 + P_i a^2 = 0$$

$$\sigma_{zz} = \frac{P_i a^2 - P_o b^2}{b^2 - a^2}$$

(Valid away from the ends !)

PRESS and SHRINK FIT PROBLEMS



A contact pressure P is created at $r=b$. $\delta = b^+ - b^-$ and $b^+ > b > b^-$

Inner cylinder outer circumference reduces to $2\pi b$ from $2\pi b^+$.

Outer cylinder inner circumference expands to $2\pi b$ from $2\pi b^-$.

$$\left(\varepsilon_{\theta\theta}\right)_i = \frac{2\pi b - 2\pi b^+}{2\pi b^+} = \frac{b-b^+}{b^+} \approx \frac{b-b^+}{b} = \frac{\delta_i}{b}$$

$$\left(\varepsilon_{\theta\theta}\right)_o = \frac{b-b^-}{b^-} \approx \frac{b-b^-}{b} = \frac{\delta_o}{b}$$

Initial Interference, $\delta = b^+ - b^- = -(b-b^+) + (b-b^-) = \delta_o - \delta_i$ So, $\delta = \delta_o - \delta_i$.

$\delta = b\left(\varepsilon_{\theta\theta}\right)_o - b\left(\varepsilon_{\theta\theta}\right)_i$ At this point assume $\sigma_{zz}=0$.

$$\delta = b\left(\frac{1}{E_o}\left\{\left(\sigma_\theta\right)_o - \nu_o\left(\sigma_r\right)_o\right\}\right) - b\left(\frac{1}{E_i}\left\{\left(\sigma_\theta\right)_i - \nu_i\left(\sigma_r\right)_i\right\}\right) \quad \text{........... (1)}$$

<u>For the outer cylinder, at *r*=*b* :</u> $(\sigma_\theta)_o = \dfrac{c^2+b^2}{c^2-b^2}P$ and $(\sigma_r)_o = -P$ ....(2)

<u>For the inner cylinder, at *r*=*b* :</u> $(\sigma_\theta)_i = -\dfrac{b^2+a^2}{b^2-a^2}P$ and $(\sigma_r)_i = -P$ ....(3)

Substituting (2) and (3) in (1),

$$\delta = Pb\left\{\frac{1}{E_o}\left(\frac{c^2+b^2}{c^2-b^2}+\nu_o\right)+\frac{1}{E_i}\left(\frac{b^2+a^2}{b^2-a^2}-\nu_i\right)\right\}$$

given δ, P can be found and vice versa.

It is assumed that members are of the same length, otherwise stress concentration occurs at the ends.