Static Design Criteria
Failure Theories

Overview

- In case of uniaxial loading, there is only one value of stress, and one value of strength.

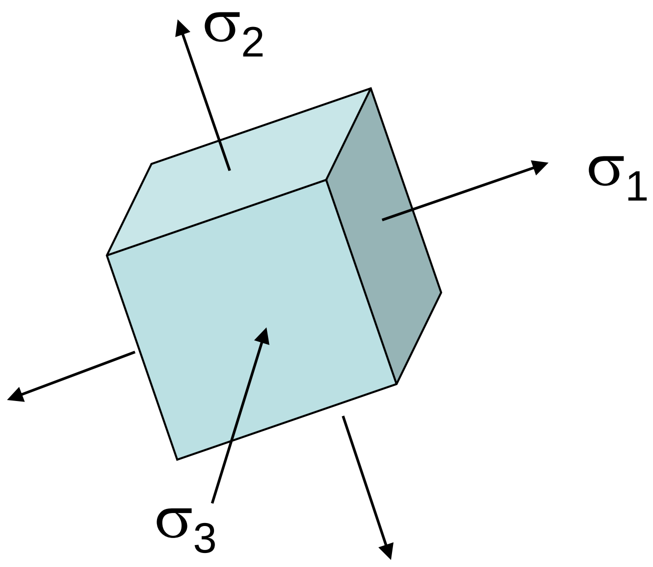
- Stress : σ=F/A Strength: S_y
- What other types of uniaxial loading do we have?
- It is easy to evaluate safety. $n=S_y/\sigma$
- What do we do in case of multi-axial loading to evaluate safety?

Overview

- We use Failure Theories
- The Failure theories that we are going to consider are as follows:
  - Maximum Normal Stress Theory
  - Maximum Shear Stress Theory
  - Distortion Energy Theory
  - Coulomb Mohr Theory
  - Modified Mohr Theory

Maximum Normal Stress Theory

- According to MNST failure occurs whenever one of the three principal stresses equals the strength.



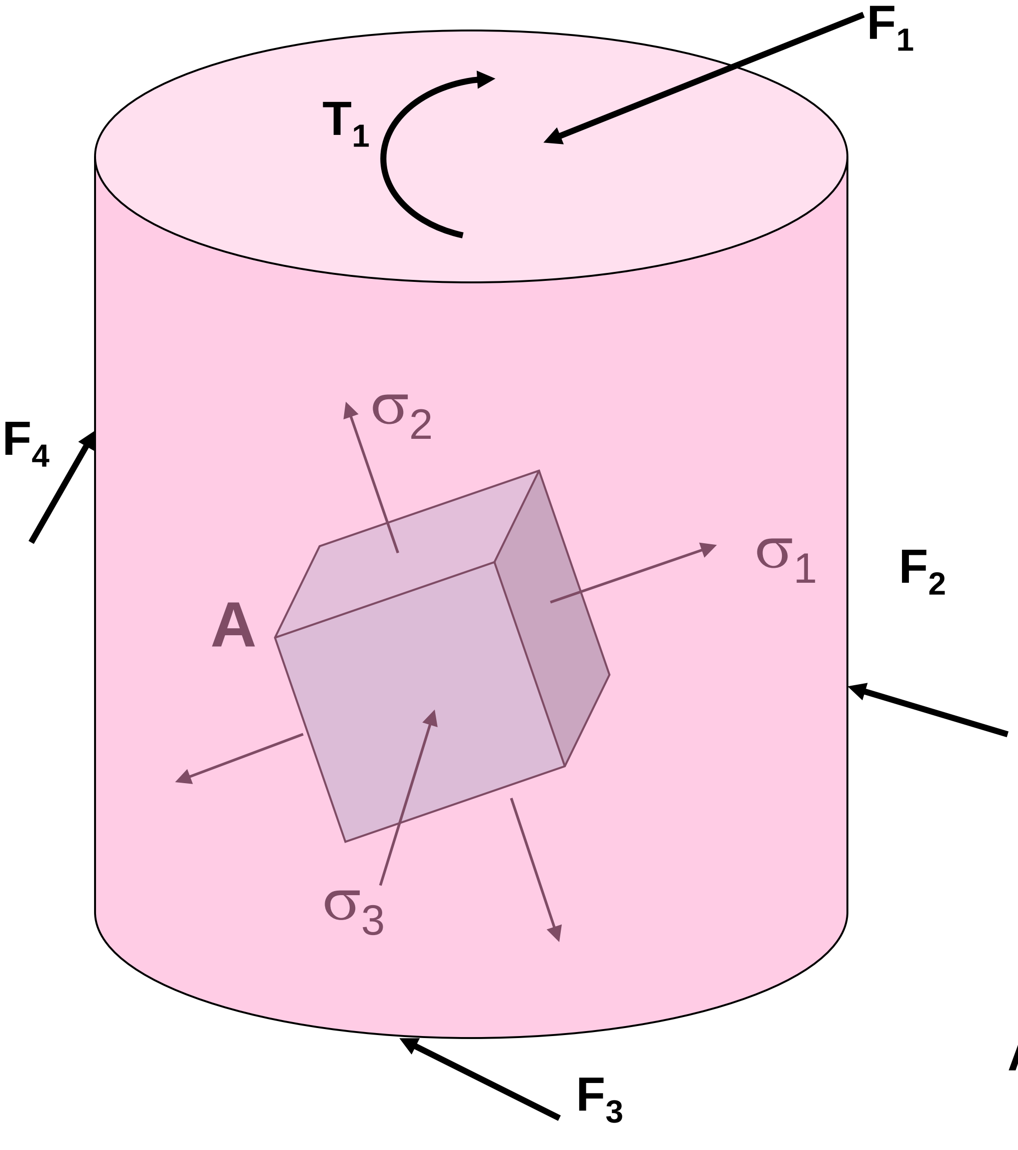
Maximum Normal Stress Theory

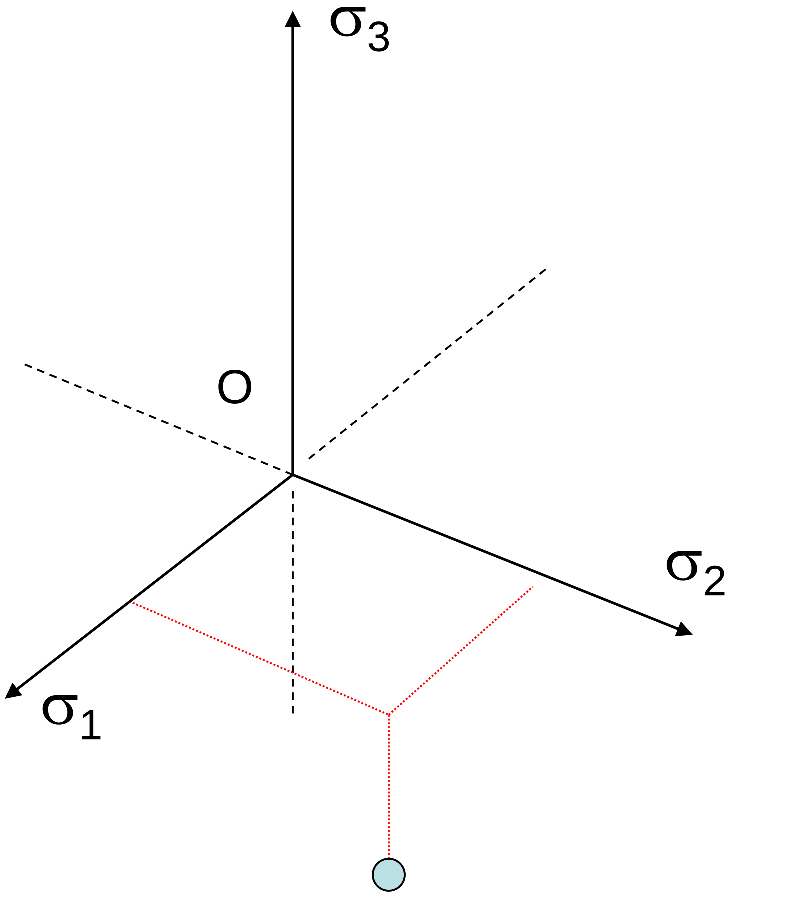
- Let $\sigma_1 > \sigma_2 > \sigma_3$
- According to MNST failure occurs whenever
- $\sigma_1 = S_t$ or $\sigma_3 = -S_c$
- S_t and S_c are tensile and compressive strengths, respectively.
- MNST is usually applied to brittle failure.
- In this case S_t and S_c are ultimate strengths.

Maximum Normal Stress Theory

- In case MNST is applied to a ductile failure S_t and S_c are yield strengths.
- Safe combinations of σ_1 , σ_2 , σ_3 can be represented geometrically in the so called "stress space".
- In the stress space, the principal stresses (σ_1 , σ_2 , σ_3) at a given point in the material represent the coordinates of a point.

Maximum Normal Stress Theory

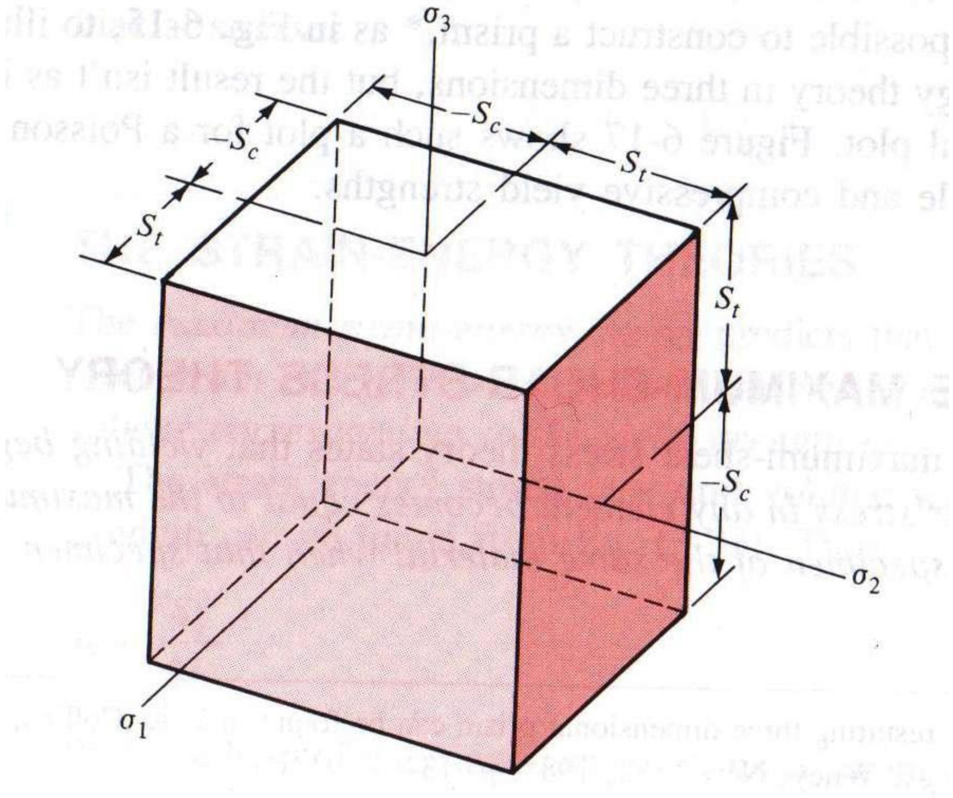




A is a point in a machine element

Maximum Normal Stress Theory

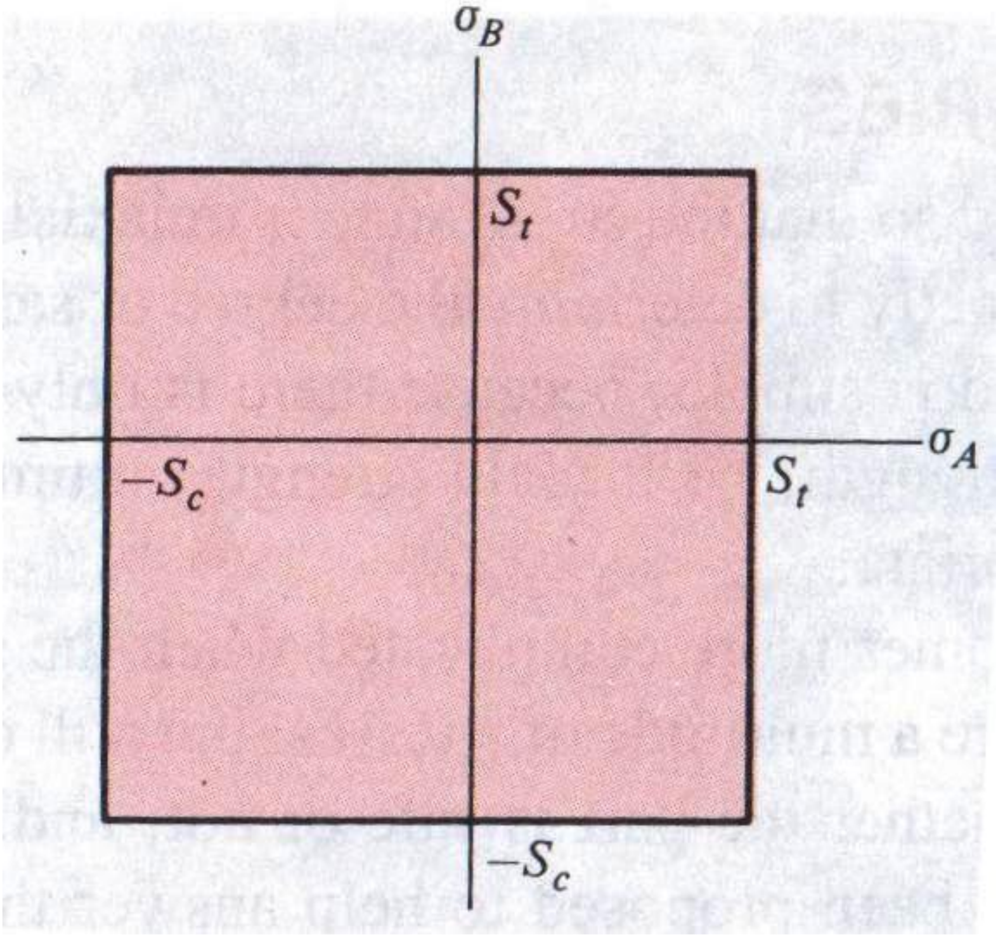
- According to MNST safe combinations of σ_1 , σ_2 , σ_3 are inside the cube shown.



Maximum Normal Stress Theory

- S_t and S_c need not be equal
- In case of biaxial loading (where one of the principal stresses is always zero), safe combinations of principal stresses (σ_A,σ_B) are inside the square shown in 2D stress space.
- When do we have one of the principal stresses is always zero?

Maximum Normal Stress Theory

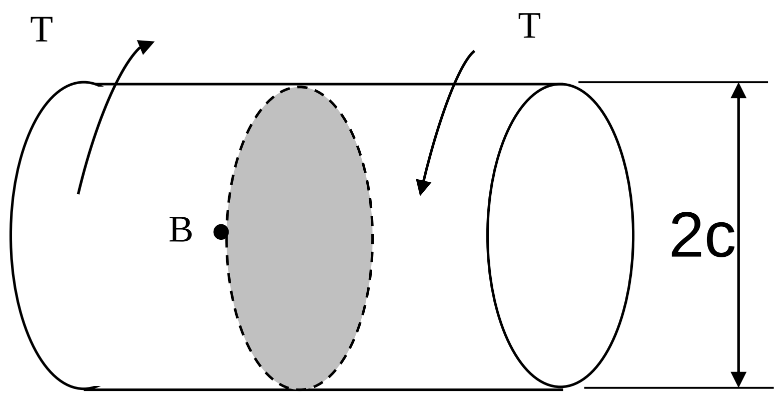


Maximum Normal Stress Theory

- MNST is very easy to use, unfortunately it has a serious shortcoming for ductile materials.
- For ductile materials, yield strengths in tension and compression are usually the same. $S_{yt}=S_{yc}=S_y$.
- Prediction of MNST may become uncon-servative if the largest and the smallest principal stresses have opposite signs.

Maximum Normal Stress Theory

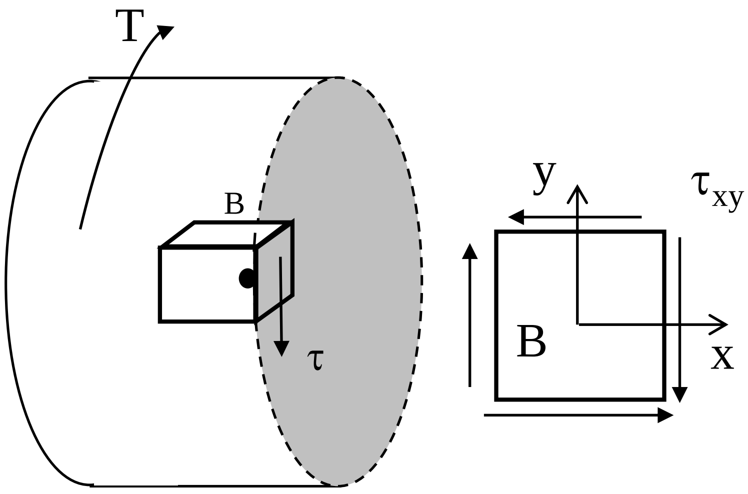
- To see this consider the case of pure torsion
- Take a section across point B which is on the surface.

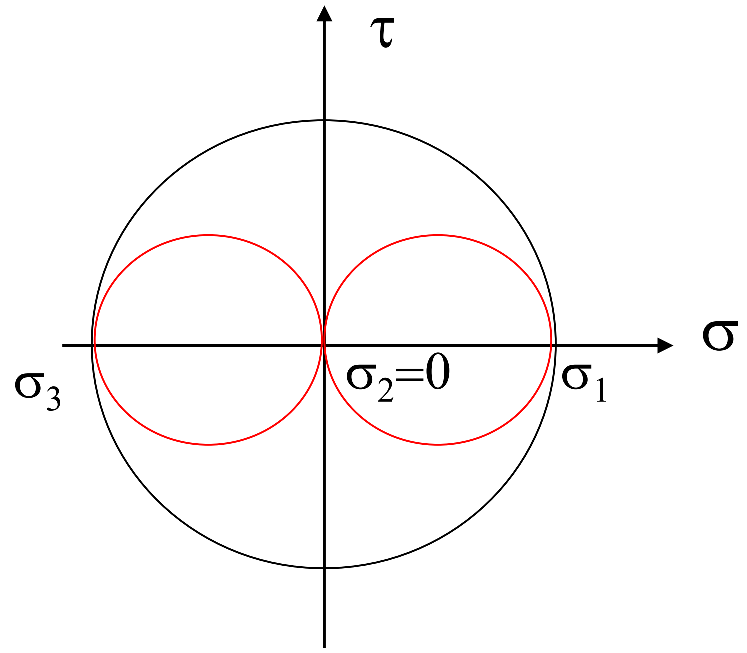


Maximum Normal Stress Theory

τ_{xy} is the only nonzero stress.

For a point B on the surface,
3D Mohr circle is as follows:





$$\sigma_1 = -\sigma_3 = \tau_{xy} = T.c/J$$

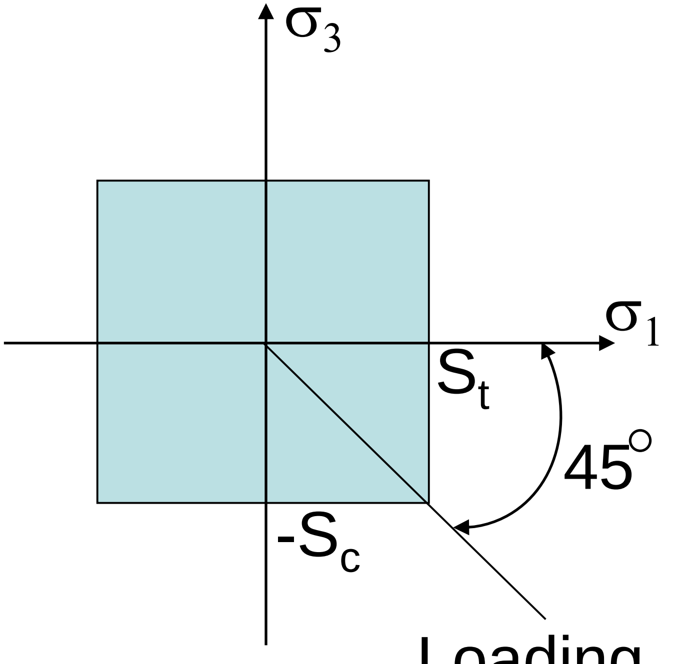
Maximum Normal Stress Theory

- As the applied torque T increases, pricipal stresses describe a straight line in stress space whose equation is:

$$\sigma_1 = -\sigma_3$$

- This line is called the "loading line".

Let $-S_c = S_t = S_y$



Maximum Normal Stress Theory

- So, MNST predicts that failure (onset of yielding) will occur when,
- $\sigma_1 = S_y$ where S_y is yield strength in uniaxial tension,
- But at the same time this implies, at failure
- $\tau_{xy} = S_y$, since $\tau_{xy} = \sigma_1$!
- This prediction does not agree with experiments.

Maximum Normal Stress Theory

- Yield strength under pure shear, S_{sy} can be measured with torsion experiments.
- These experiments show that for ductile materials, S_{sy} is about 60% of S_y whereas MNST predicted that they were equal.
- For this reason MNST is not recommended for ductile materials.

Maximum Normal Stress Theory

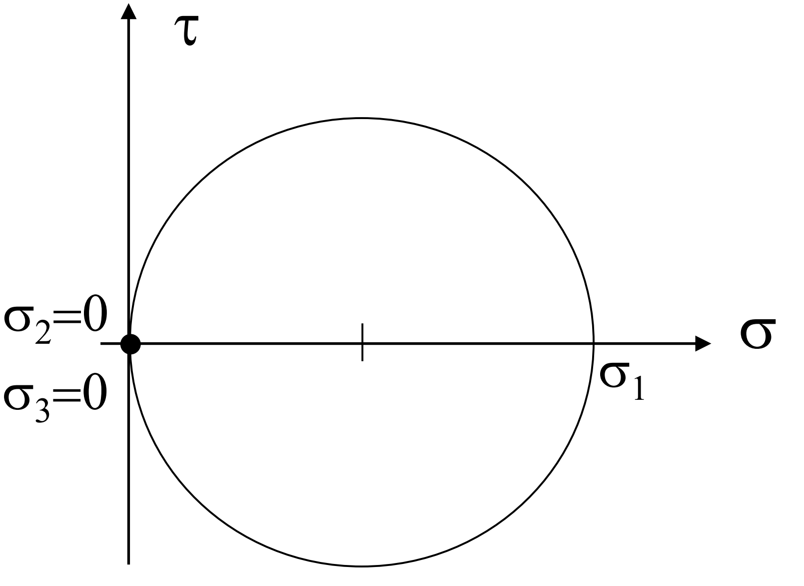
- On the other hand for brittle materials, it is observed from experiments that Ultimate Tensile Strength is roughly equal to Ultimate Shear Strength.
- $S_{ut}=S_{su}$
- So MNST can be used for brittle materials

Maximum Shear Stress Theory

- This theory is also known as Tresca Yield Condition.
- It is proposed to predict yielding, so it is applicable to ductile materials.
- According to MSST, yielding begins when-ever the max. shear stress in any mechan-ical element becomes equal to the max. shear stress in a tension test specimen of the same material when that specimen begins to yield.

Maximum Shear Stress Theory

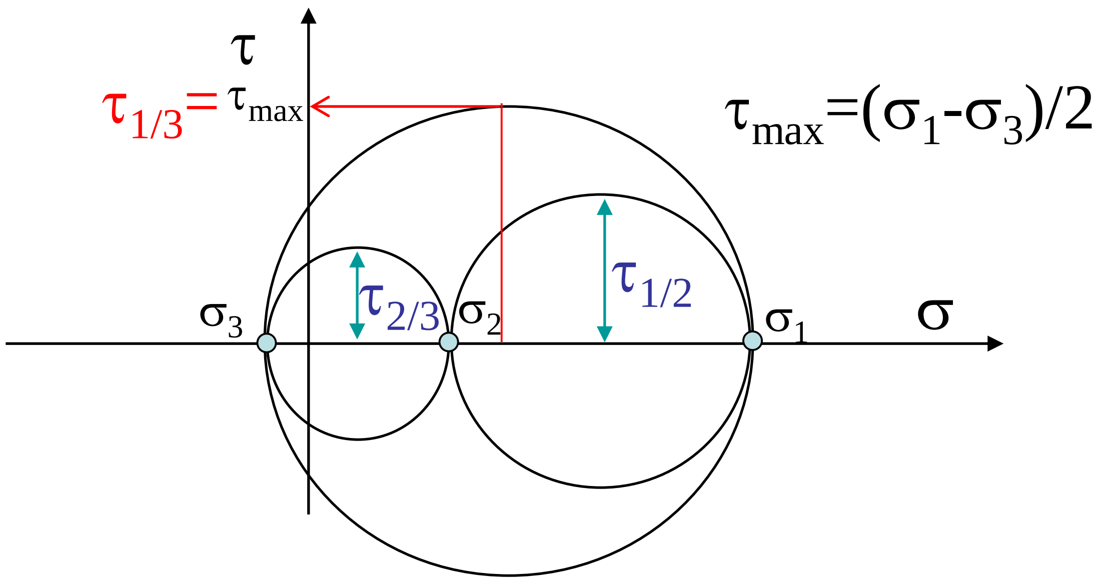
- For uniaxial tension test, from the 3D Mohr circle, one can observe that the maximum shear stress is;
- $\tau_{max} = \sigma_1/2 = S_y/2$



Maximum Shear Stress Theory

- Therefore this theory implies that yield strength in shear, S_{sy} is half of the tensile yield strength.
- $S_{sy}=S_y/2$
- For a general 3D stress state, let the principal stresses be;
- $\sigma_1 > \sigma_2 > \sigma_3$

Maximum Shear Stress Theory



- According to MSST, yielding will occur as soon as $\tau_{max} \geq Ssy = Sy/2$
- In other words, $\sigma_1 - \sigma_3 \geq S_y$

Maximum Shear Stress Theory

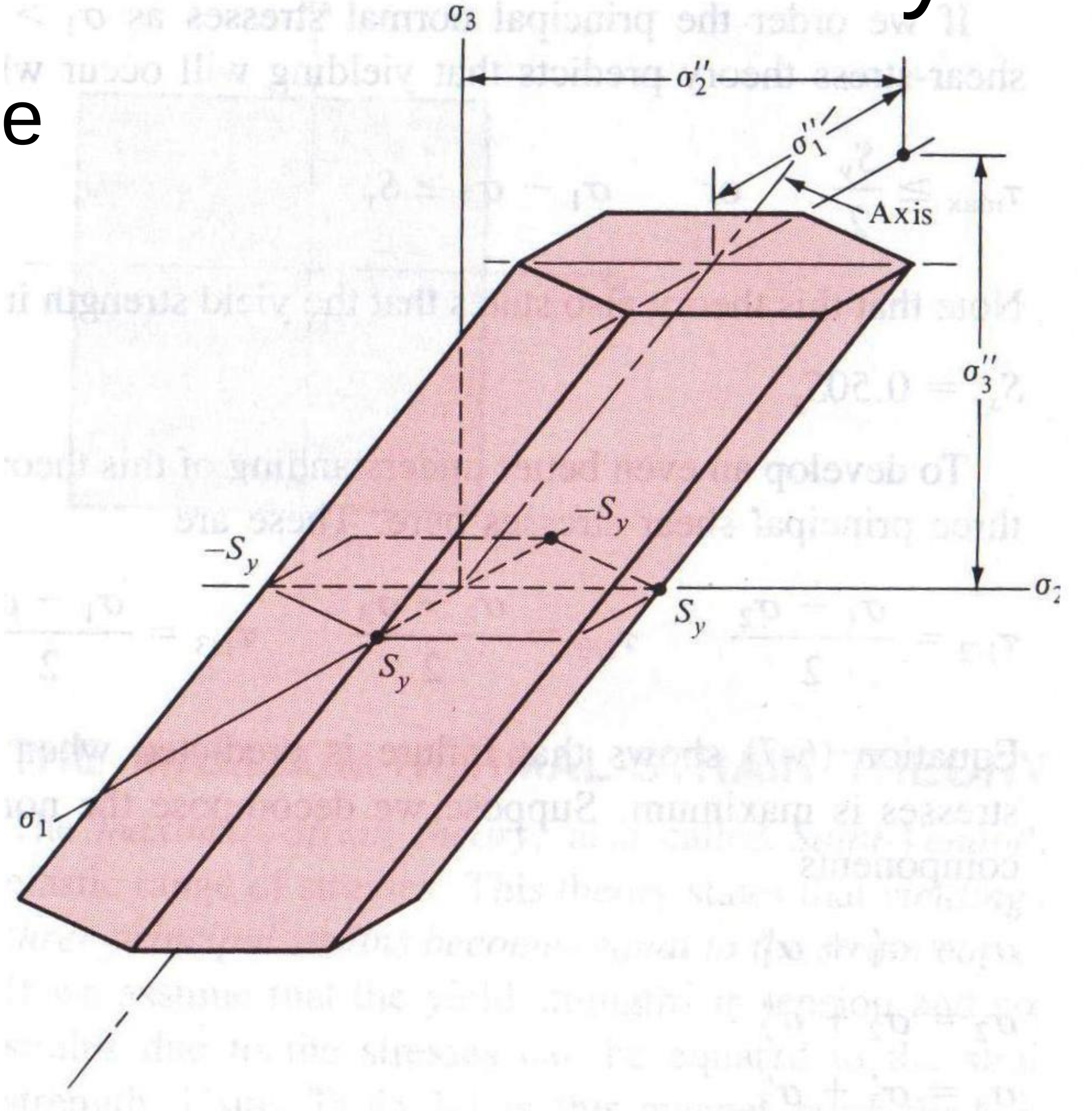
- In general maximum shear stresses in 12, 23 and 13 planes are given by
- $\tau_{1/2} = (\sigma_1 - \sigma_2)/2$,
- $\tau_{2/3} = (\sigma_2 - \sigma_3)/2$,
- $\tau_{1/3} = (\sigma_1 - \sigma_3)/2$
- As soon as one of these shear stresses becomes equal to $S_{sy}=S_y/2$, yielding begins.
- Sign of the shear stress indicates their direction.

Maximum Shear Stress Theory

- Then the following six equations define 6 planes. The volume inside these planes is the safe region in 3D stress space.
- $\sigma_1 - \sigma_2 = \pm S_y$
- $\sigma_2 - \sigma_3 = \pm S_y$
- $\sigma_1 - \sigma_3 = \pm S_y$
- Planes form an obligue regular hexagonal prism with hydrostatic line as its axis.

Maximum Shear Stress Theory

- Hydrostatic line
- $\sigma_1 = \sigma_2 = \sigma_3$



Maximum Shear Stress Theory

- Note that we can write stresses as follows:

$$\sigma_1 = \sigma_1' + \sigma_1''$$
$$\sigma_2 = \sigma_2' + \sigma_2''$$
$$\sigma_3 = \sigma_3' + \sigma_3''$$

where

$$\sigma_1' = (2\sigma_1 - \sigma_2 - \sigma_3)/3$$
$$\sigma_2' = (2\sigma_2 - \sigma_1 - \sigma_3)/3$$
$$\sigma_3' = (2\sigma_3 - \sigma_1 - \sigma_2)/3$$

and

$$\sigma_1'' = \sigma_2'' = \sigma_3'' = (\sigma_1 + \sigma_2 + \sigma_3)/3$$

Maximum Shear Stress Theory

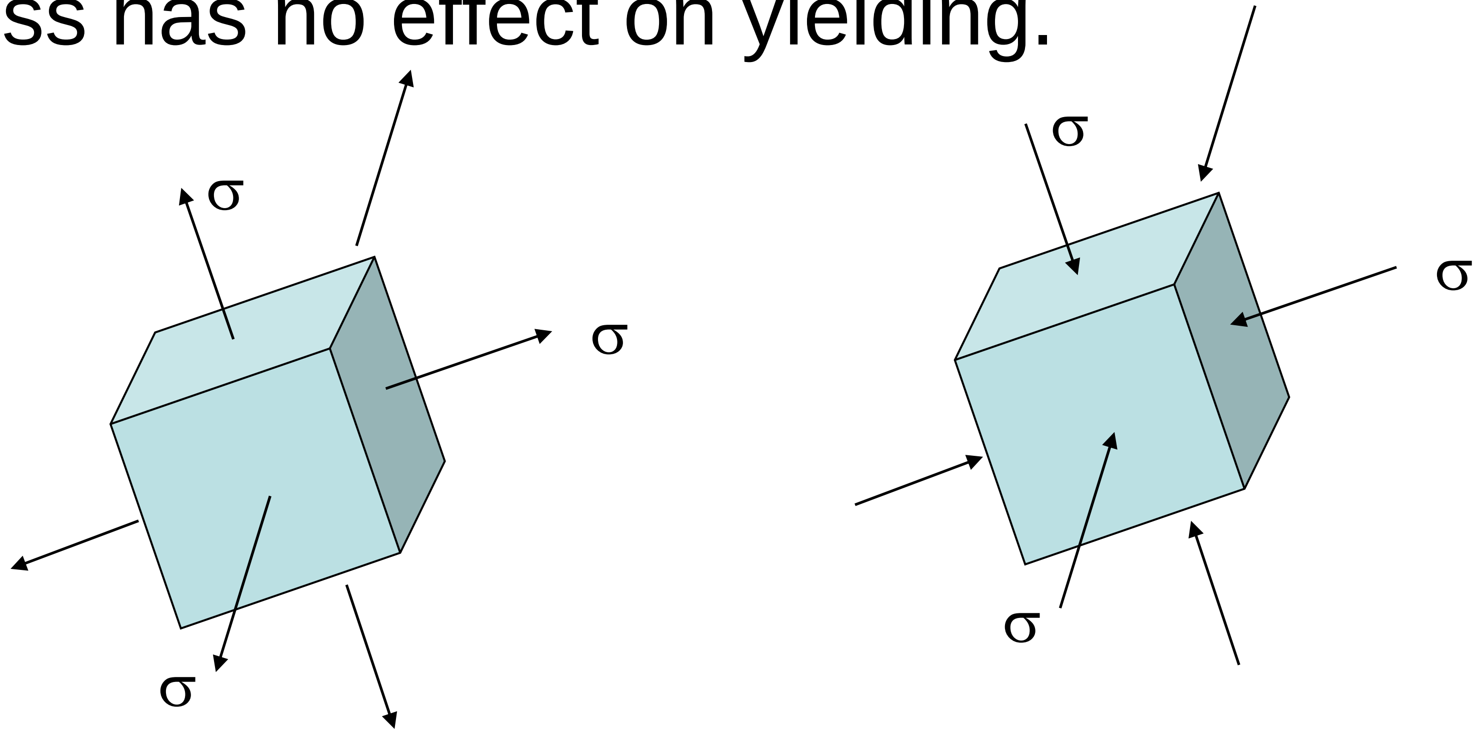
- Components with double prime are called hydrostatic components
- Components with single prime are called deviatoric components
- We can easily observe that, at yielding,

$$\sigma_1-\sigma_2=\sigma_1'-\sigma_2'=\pm S_y \quad \sigma_2-\sigma_3=\sigma_2'-\sigma_3'=\pm S_y$$

$$\sigma_1-\sigma_3=\sigma_1'-\sigma_3'=\pm S_y$$

Maximum Shear Stress Theory

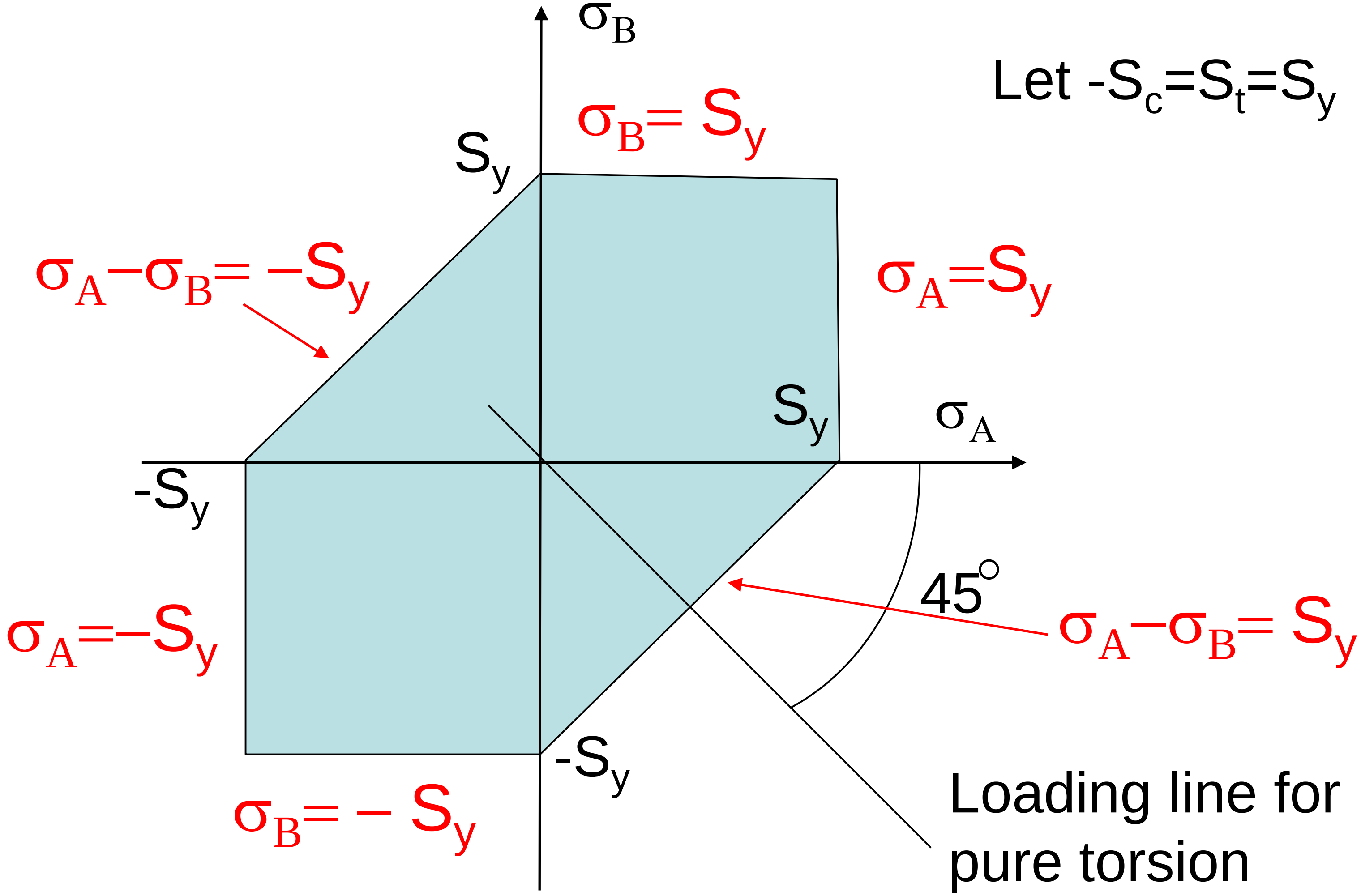
- This means that according to this theory, no matter how large it may be, hydrostatic stress has no effect on yielding.



Maximum Shear Stress Theory

- For a biaxial case let σ_3=0, and σ_A and σ_B be the non-zero principal stresses.
- In this case, the safe region in the 2D stress state is as shown below.
- Note that this the intersection of the obligue hexagonal prism with the σ_3=0 plane.

Maximum Shear Stress Theory



Maximum Shear Stress Theory

- Shaded area represents the safe region. (i.e. safe combinations of σ_A and σ_B)
- These σ_A and σ_B satisfy all of the six inequalities.

$$\sigma_A - \sigma_B \leq S_y \qquad \sigma_B \leq S_y \qquad \sigma_A \leq S_y$$

$$\sigma_A - \sigma_B \geq -S_y \qquad \sigma_B \geq -S_y \qquad \sigma_A \geq -S_y$$

Maximum Shear Stress Theory

- Note that in the 1st and the 3rd quadrant MNST and MSST make the same prediction.
- 2nd and 4th zones where principal stresses have opposite sign differ.
- In design problems an effective (equivalent) stress can be defined as follows:

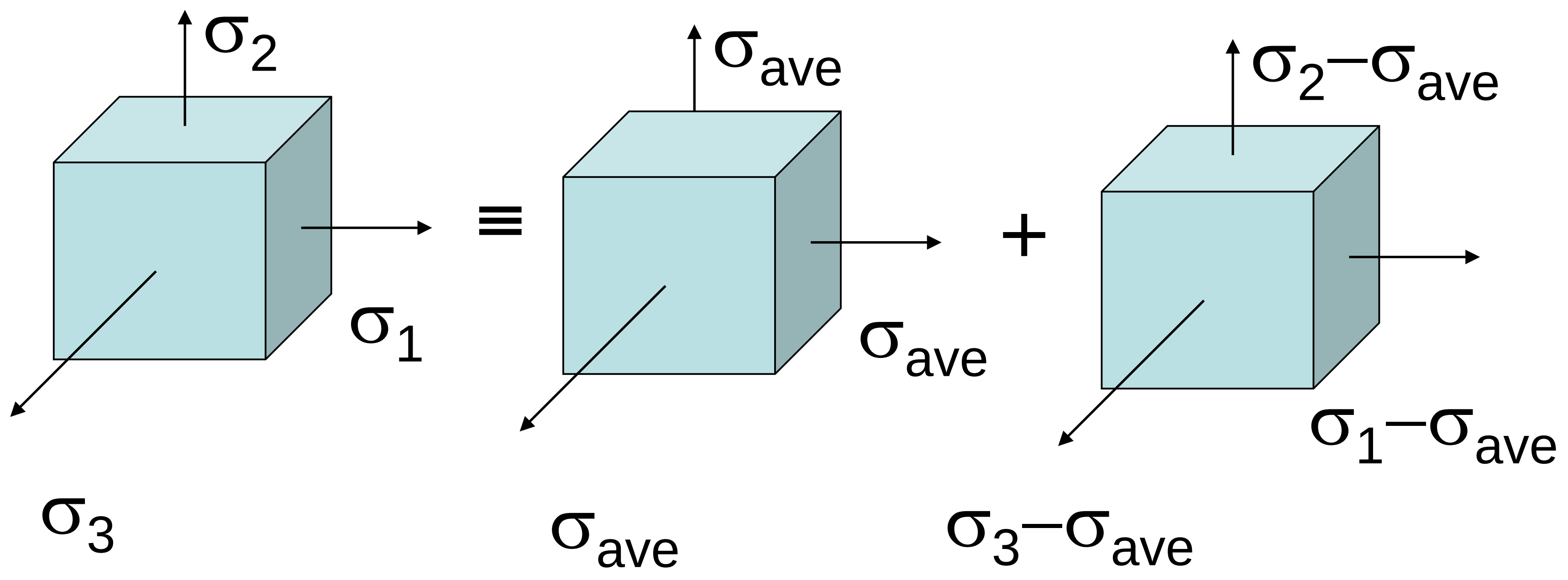
Maximum Shear Stress Theory

- Let $\sigma_1 > \sigma_2 > \sigma_3$
- $\sigma' = \sigma_1 - \sigma_3$
- According to MSST, yielding will occur as soon as $\sigma' \geq S_y$
- Then factor of safety, $n = S_y / \sigma'$

Distortion Energy Theory

- Strain energy stored in a unit volume of element due to combined loading can be split up into two parts.
- Strain energy associated with the volume change
- Strain energy associated with distortion

Distortion Energy Theory



- Let $\sigma_1 > \sigma_2 > \sigma_3$
- $\sigma_{ave} = (\sigma_1 + \sigma_2 + \sigma_3)/3$

Distortion Energy Theory

- Recall that Strain Energy per unit volume is given in terms of principal stresses as follows:

$$u = \frac{1}{2E}\left(\sigma_1{}^2 + \sigma_2{}^2 + \sigma_3{}^2 - 2\nu\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3\right)\right)$$

- If we substitute $\sigma_1=\sigma_2=\sigma_3=\sigma_{ave}$ in the equation above, we get strain energy per unit volume associated with volume change.

Distortion Energy Theory

$$u_v = \frac{3\sigma_{ave}^{\ \ 2}}{2E}(1-2\nu)$$

- than strain energy per unit volume associated with distortion (DE) is given as follows:

$$u_d = u - u_v = \frac{1+\nu}{3E}\left(\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}\right)$$

- Note that DE is zero for hydrostatic stress state ($\sigma_1=\sigma_2=\sigma_3$).

Distortion Energy Theory

- DET predicts that yielding will occur whenever the DE equals the distortion energy in the same volume, when it is uniaxially stressed to the yield strength.
- For simple tension test, let $\sigma_1=\sigma'$, $\sigma_2=\sigma_3=0$ in the expression for DE:

$$\left(u_d\right)_{uni.}=\frac{1+\nu}{3E}\sigma'^2$$

Distortion Energy Theory

- Equating the DE for uniaxial case to general expression of DE and solving for σ',

$$\sigma' = \sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2}{2}}$$

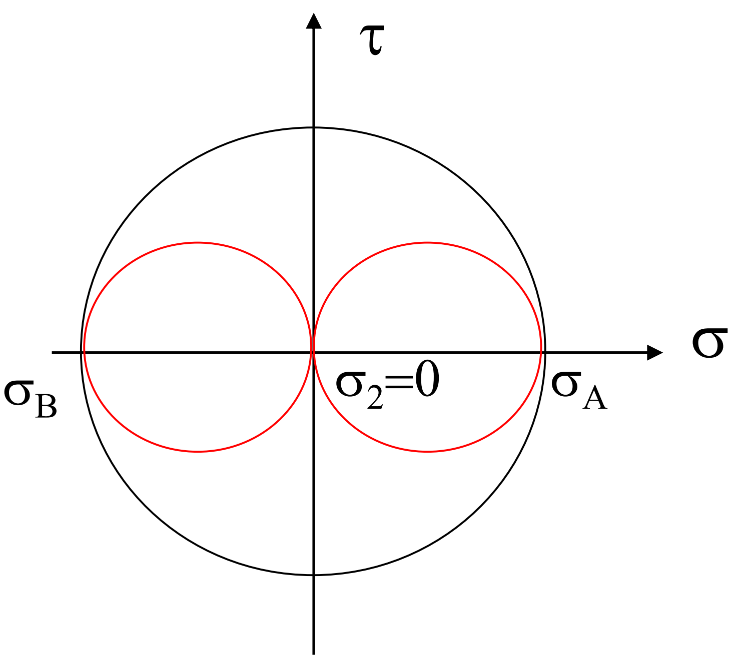
- Yielding will occur when $\sigma' = S_y$.
- σ' which is the effective stress, represen-ting the entire stress state is called Von Mises stress.

Distortion Energy Theory

- For biaxial stress state, let σ_A and σ_B be the two non zero principal stresses.
- Then Von Mises stress becomes equal to
- $\sigma'=(\sigma_A^2-\sigma_A\sigma_B+\sigma_B^2)^{1/2}$

For pure torsion Mohr circle becomes;

$\sigma_A=-\sigma_B=\tau_{max}=T.c/J$



Distortion Energy Theory

- Then Von Mises stress for pure torsion becomes equal to

$$\sigma' = \sqrt{\tau_{\max}{}^2 - \tau_{\max}(-\tau_{\max}) + (-\tau_{\max})^2}$$

$$\sigma' = \sqrt{3}\tau_{\max}$$

- Then yielding under pure torsion occurs when

$$\sigma' = \sqrt{3}\tau_{\max} = S_y$$

- This implies yield strength under pure shear, Ssy, according to DET is;

$$S_{sy} = \frac{S_y}{\sqrt{3}} \cong 0.577 S_y$$

Distortion Energy Theory

- The result $S_{sy}=0.577S_y$ agrees very well with experiments.
- The DET is also called
  - The octahedral shear stress theory
  - The Von Mises - Hencky theory
- From

$$\sigma' = \sqrt{\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_1-\sigma_3)^2}{2}} = S_y$$

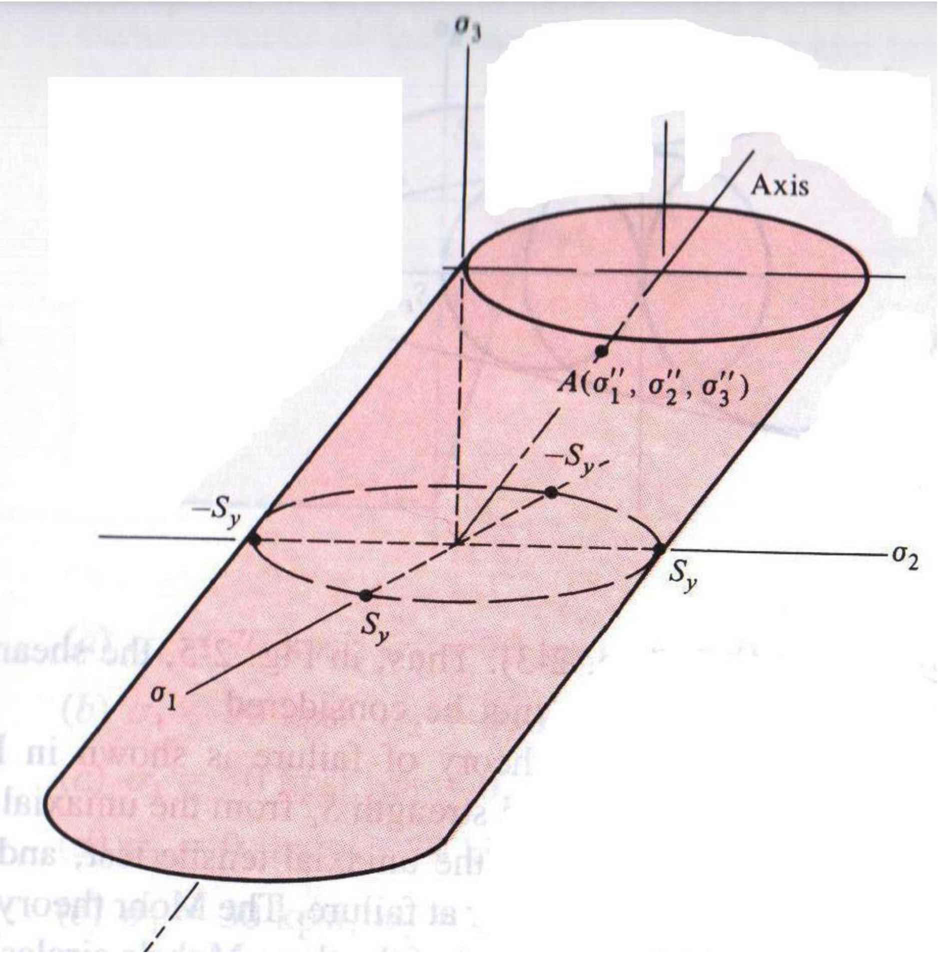
Distortion Energy Theory

- The equation of the surface enclosing the safe region in stress space is:

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2 = 2S_y{}^2$$

- This equation describes an obligue elliptical cylinder with hydrostatic line as its axis.
- Safe combinations of principal stresses are inside this cylinder.

Distortion Energy Theory



Distortion Energy Theory

- Instead of principal stresses, one can calculate Von Mises stress in terms of a given stress state.

$$\sigma' = \sqrt{\frac{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_x - \sigma_z)^2 + 6(\tau_{xy}{}^2 + \tau_{yz}{}^2 + \tau_{xz}{}^2)}{2}}$$

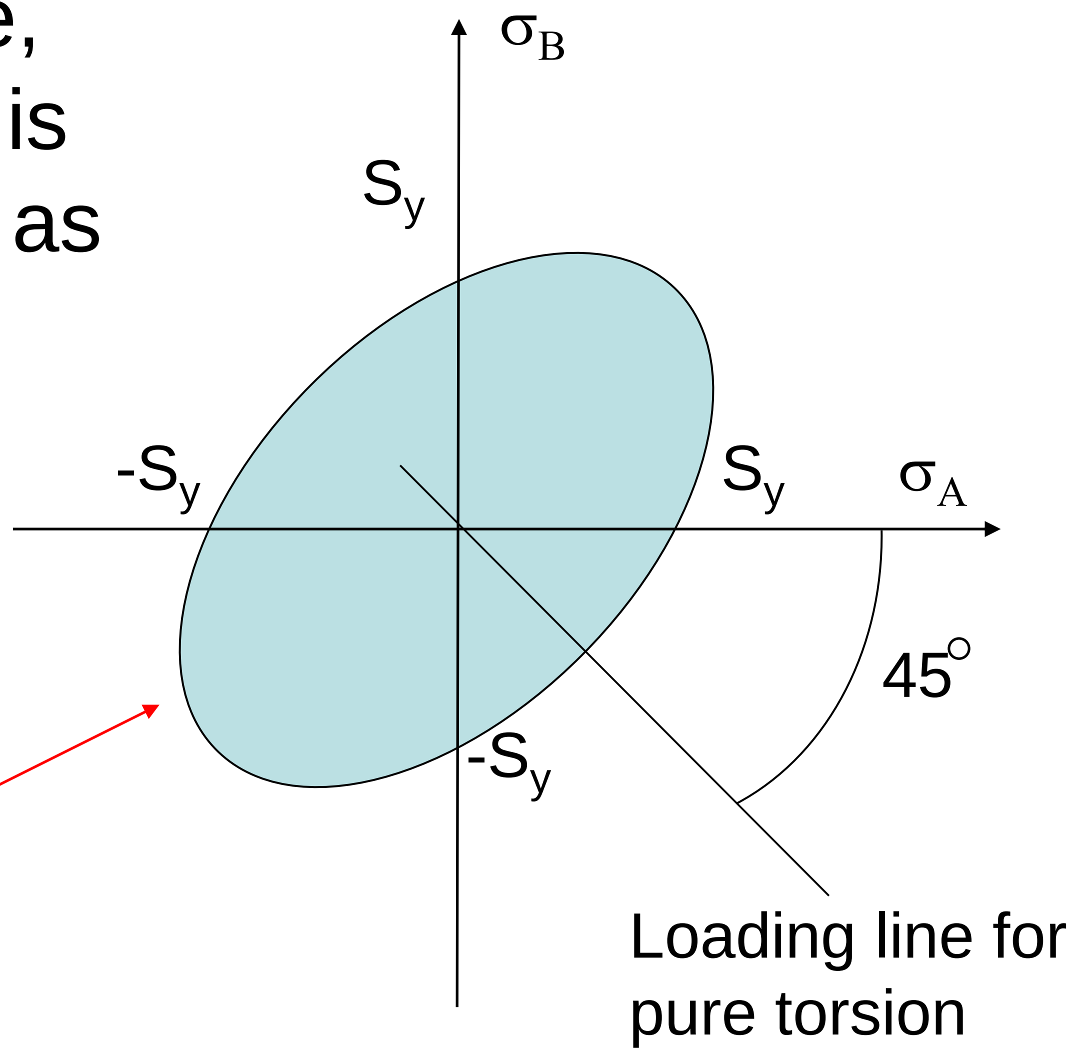
- Safety Factor can be calculated as;

$$n = \frac{S_y}{\sigma'}$$

Distortion Energy Theory

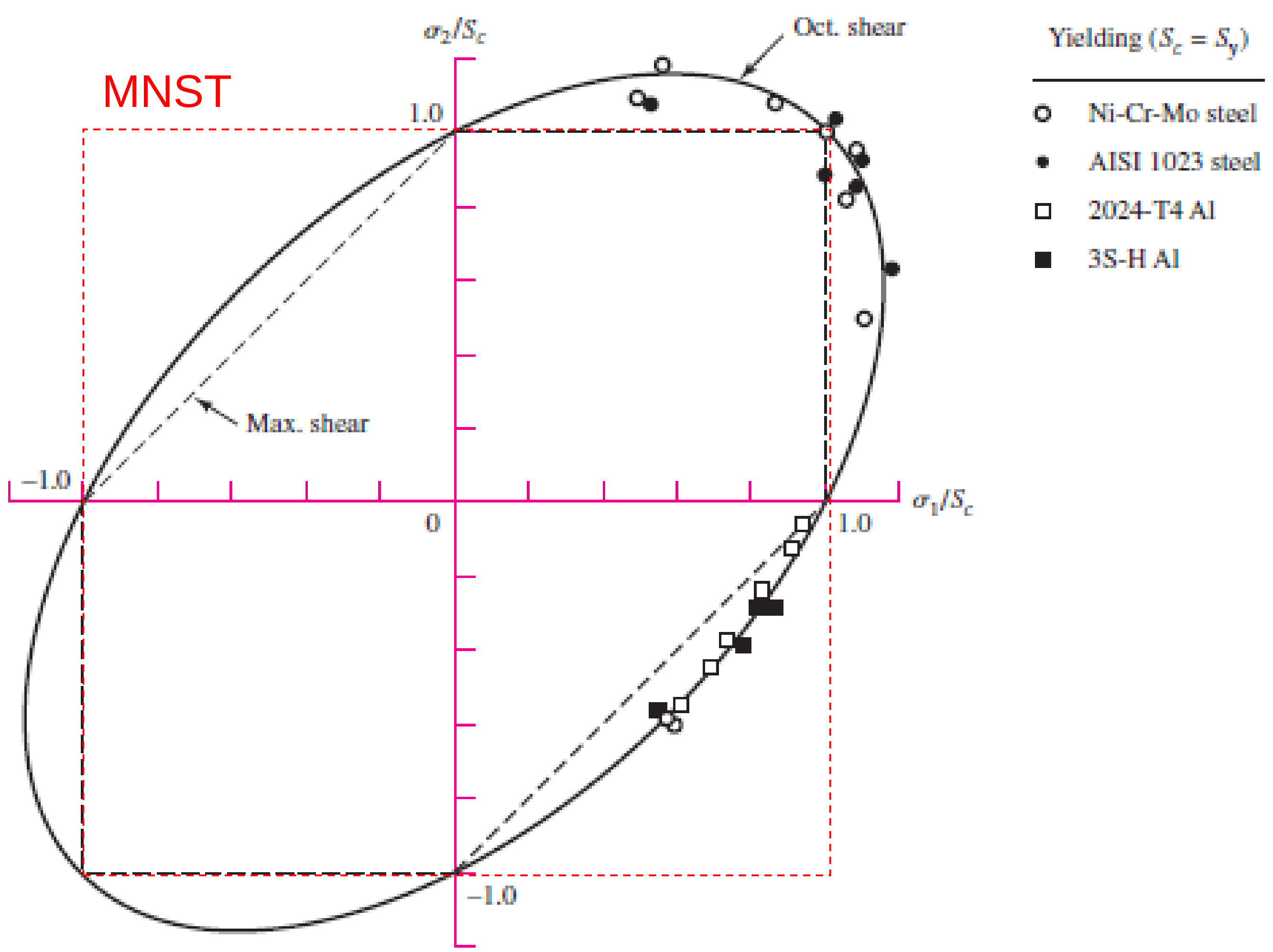
- For biaxial case, the safe region is an elliptical are as shown.

Let $-S_c = S_t = S_y$



$$\sigma_A^2 - \sigma_A \sigma_B + \sigma_B^2 = S_y^2$$

Comparison of Failure Criteria

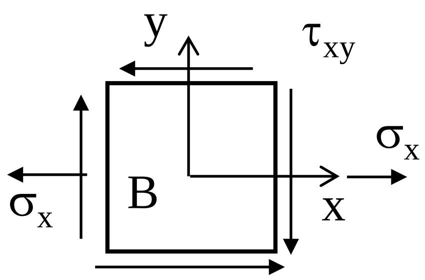


Comparison of Failure Criteria

- Note that MNST is actually useful for brittle materials.
- MSST and DET are useful for ductile materials (for predicting onset of yielding).
- MSST is more conservative than DET.
- Agreement of DET with experiments is better.

Comparison of Failure Criteria

- A common type of loading is combined bending and torsion.



- In this case principal stresses are given as

$$\sigma_A, \sigma_B = \frac{\sigma_x}{2} \pm \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^{\ 2}}$$

- Effective stresses are given as

MSST:

$$\sigma' = \sigma_A - \sigma_B = 2\sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^{\ 2}}$$

DET:

$$\sigma' = \sqrt{\sigma_A^{\ 2} - \sigma_A \sigma_B + \sigma_B^{\ 2}} = \sqrt{\sigma_x^{\ 2} + 3\tau_{xy}^{\ 2}}$$