# Design of shafts

# Introduction-1

- Shaft: A rotating member, usually of circular cross section, used to transmit power or motion.
- Carries machine elements such as such as gears, pulleys, flywheels, cranks, sprockets.
- Axle:nonrotating member that carries no torque and is used to support rotating wheels, pulleys

# Introduction-2

- Spindle : A short shaft or axle
- Shafts are typically subjected to torsion and bending.

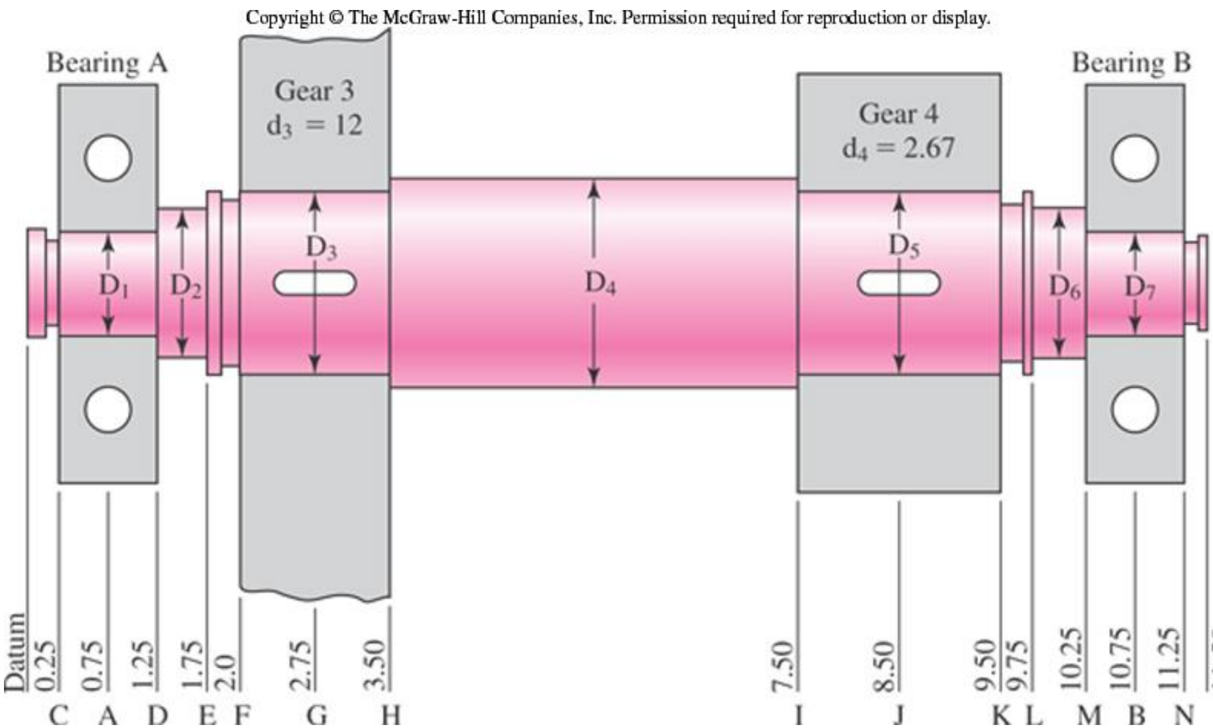
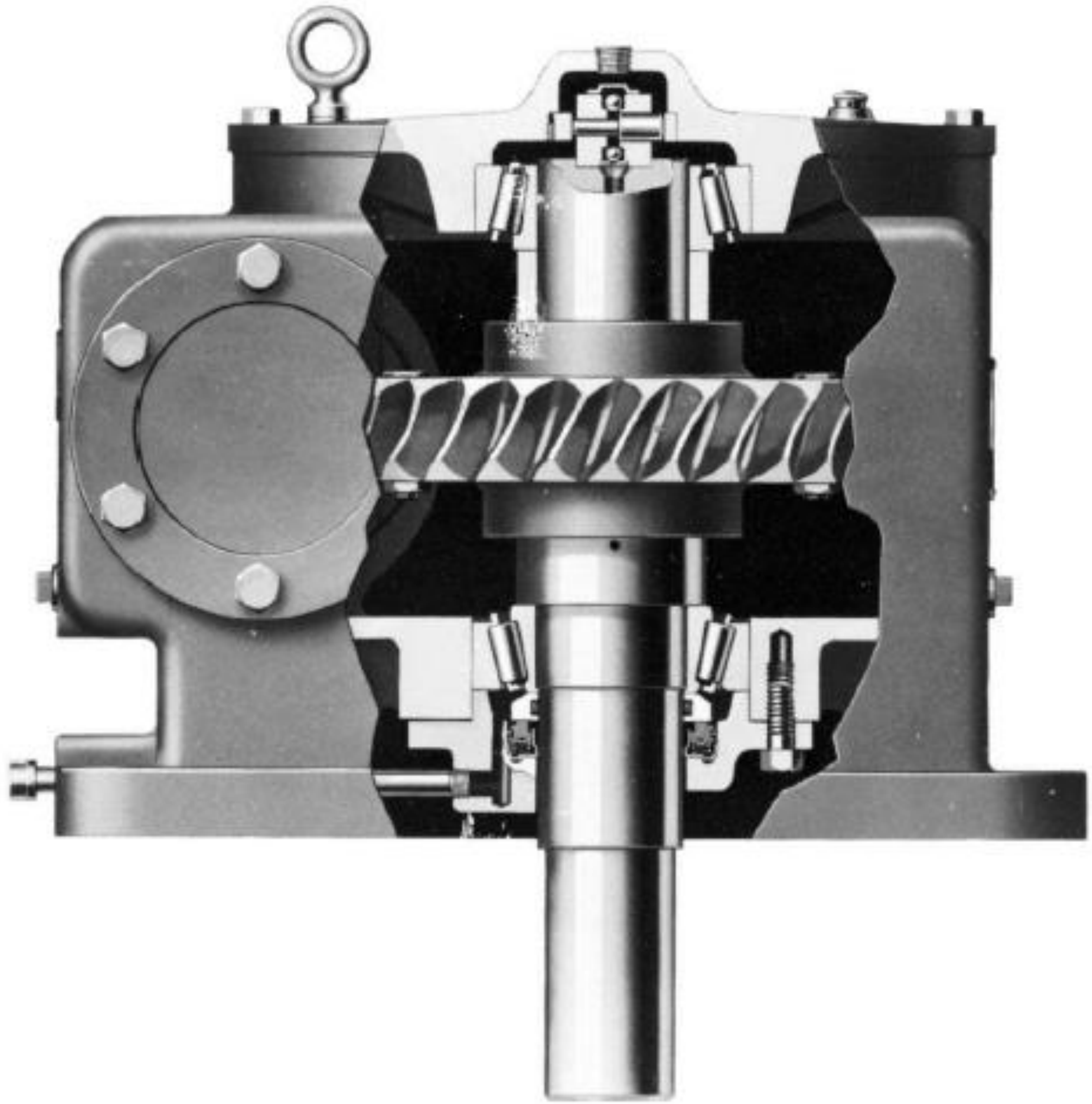


**Figure 7–1**

A vertical worm-gear speed reducer. *(Courtesy of the Cleveland Gear Company.)*



# Design Issues

- Material selection
- Geometric layout
- Stress and strength
  - Static strength
  - Fatigue strength
- Deflection and rigidity
  - Bending deflection
  - Torsional deflection
  - Slope at bearings and shaft-supported elements
  - Shear deflection due to transverse loading of short shafts
- Vibration due to natural frequency

# Shaft Materials

- Mostly mild steel
- If more strength is required alloy steels (Ni, Cr, Cr-vanadium)
  - heat treatment can be applied
- Shafts are usually produced by turning. They may be polished. Crankshafts are made by forging
- Surface hardening is usually applied.

# Power

$$P = T.\omega$$

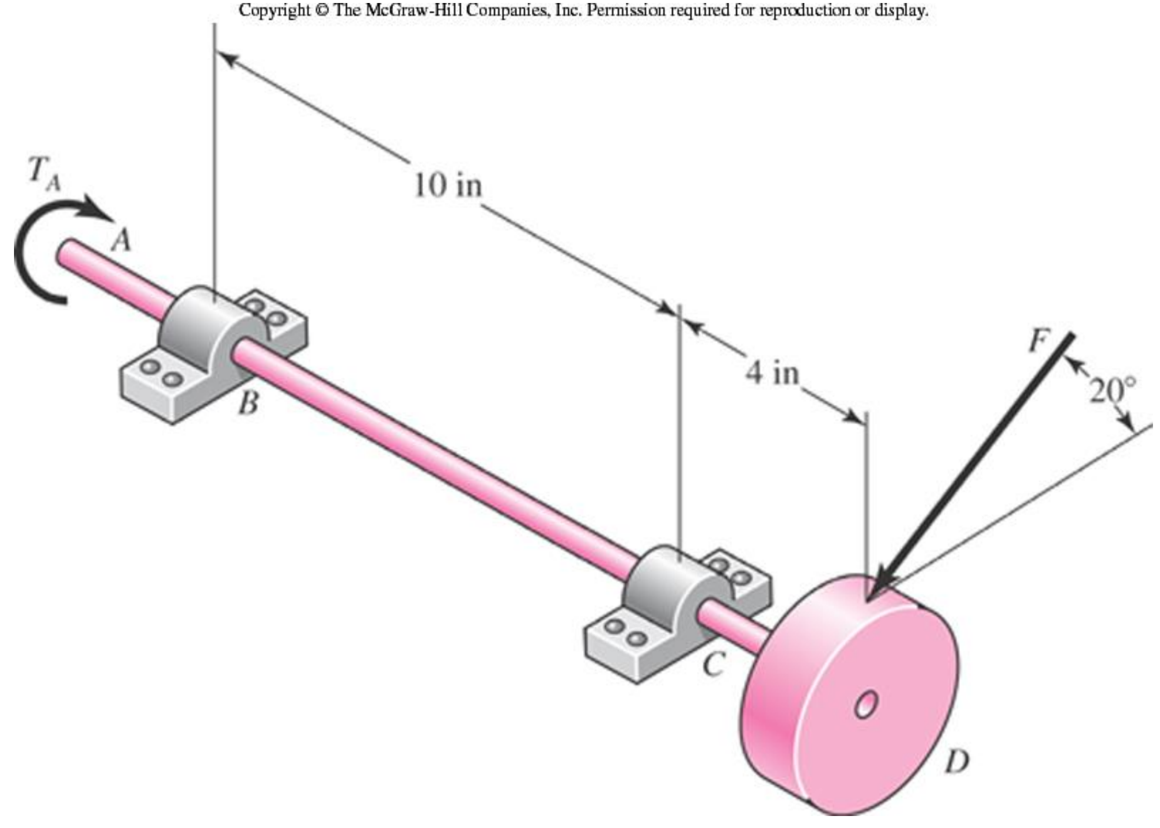
rad/s

N.m

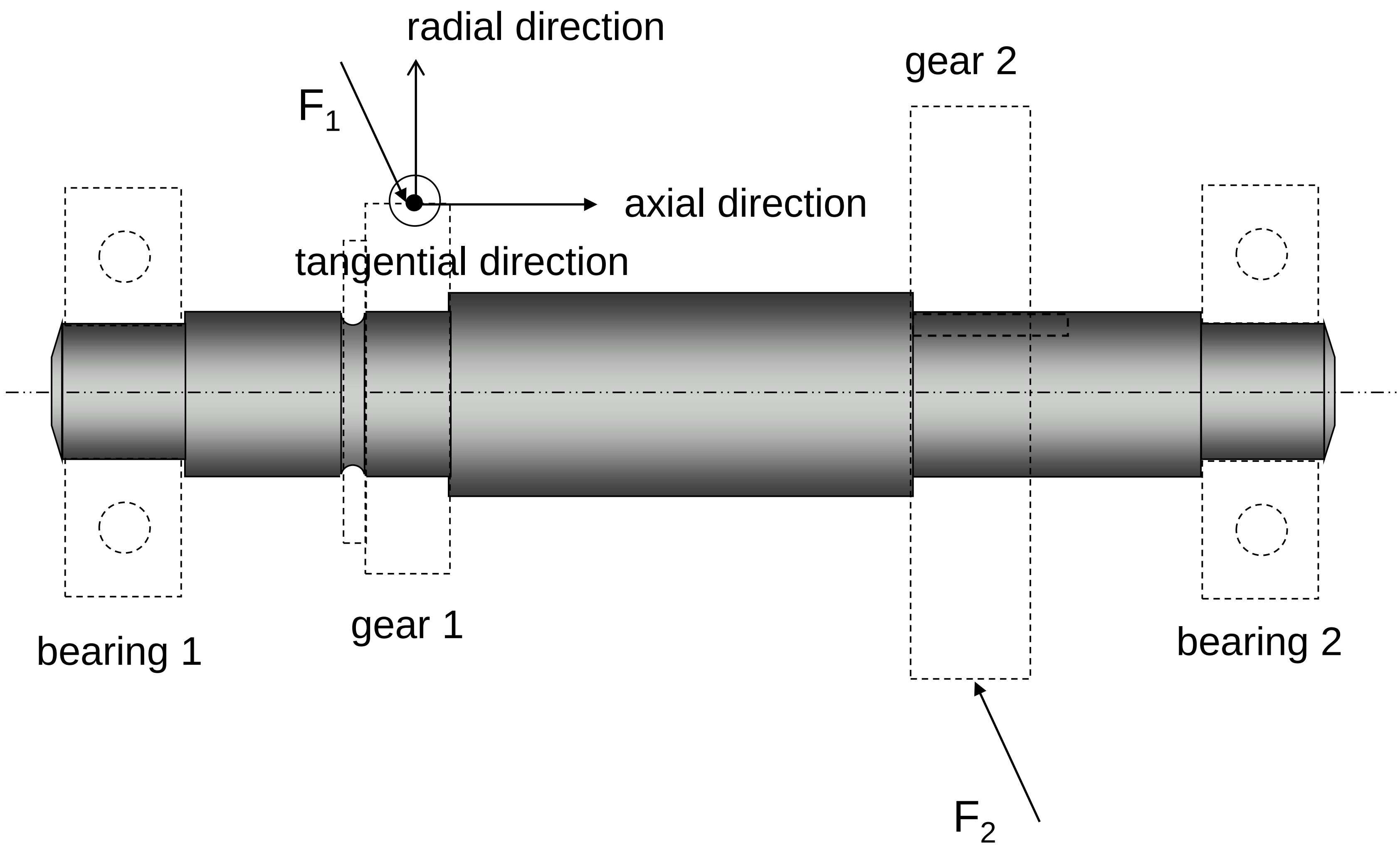
Joule/s (Watt)

1 HP=0.746 kW



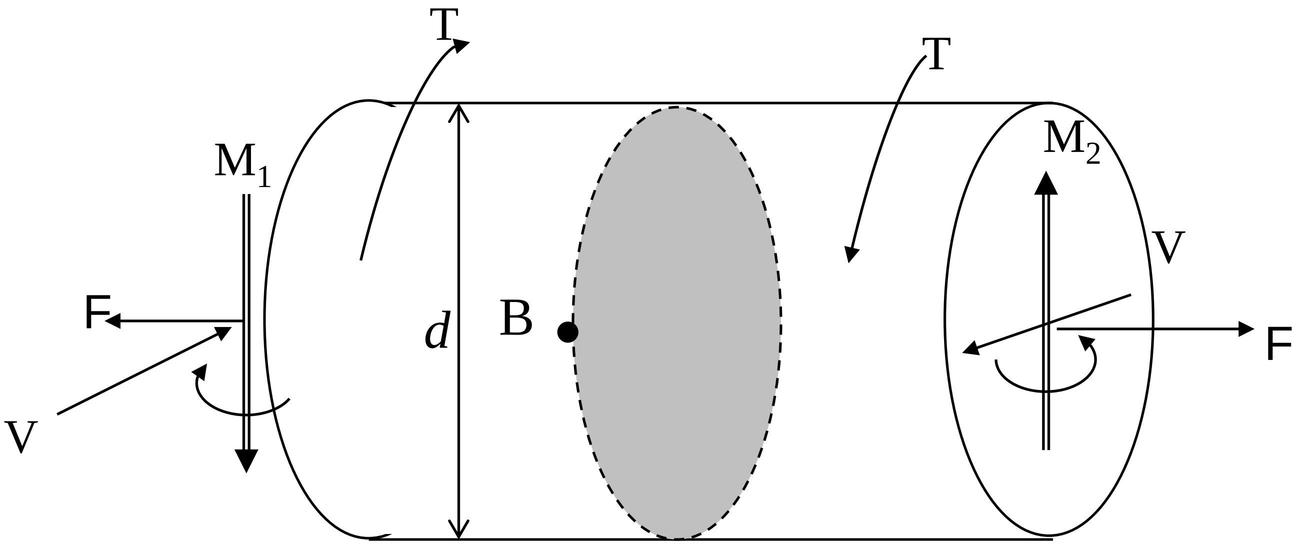


# Loads and Stresses



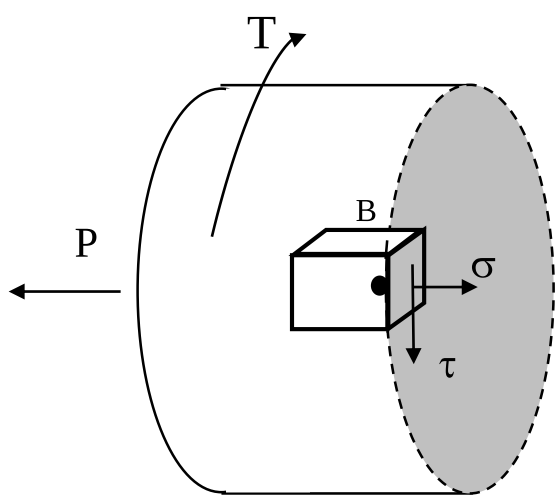
# Loads and Stresses

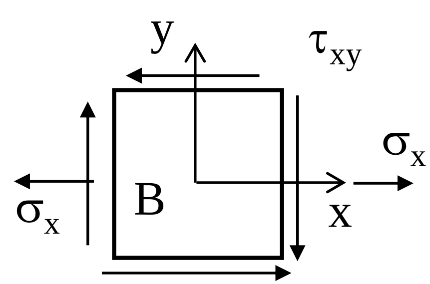
- Members on the shaft (gears, pulleys, etc.) are subjected to forces in radial, axial and tangential directions.
- These forces create torsion, bending and axial loading on the shaft. Distributions of these along the shaft should be considered to find critical sections.
- There is usually bending in two mutually perpendicular planes.




$$\sigma_x = \frac{M.c}{I} + \frac{F}{A}$$

$$\sigma_x = \frac{32M}{\pi d^3} + \frac{4F}{\pi d^2}$$




$$\tau_{xy} = \frac{T.c}{J}$$

$$\tau_{xy} = \frac{16T}{\pi d^3}$$

# Static Design of Shafts

## Maximum Shear Stress Theory

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}{}^2}$$

$$\tau_{\max} = \sqrt{\frac{1}{4}\left(\frac{32M}{\pi d^3} + \frac{4F}{\pi d^2}\right)^2 + \left(\frac{16T}{\pi d^3}\right)^2}$$

$$\tau_{\max} = \frac{16}{\pi d^3}\sqrt{\left(M + \frac{d}{8}F\right)^2 + T^2}$$

$$\tau_{\max} = \frac{S_{sy}}{n} = \frac{S_y}{2n}$$

$$\frac{S_y}{2n} = \frac{16}{\pi d^3}\sqrt{\left(M + \frac{d}{8}F\right)^2 + T^2}$$

In the presence of axial load F, diameter can not be solved in closed form.

In case F=0, diameter can be solved in closed form:

$$d = \left\{\frac{32n}{\pi S_y}\sqrt{M^2 + T^2}\right\}^{\frac{1}{3}}$$

# Static Design of Shafts

## Distortion Energy Theory

$$\sigma' = \left\{ \frac{1}{2} \left[ (\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 \right] + 3(\tau_{xy}{}^2 + \tau_{yz}{}^2 + \tau_{zx}{}^2) \right\}^{\frac{1}{2}}$$

$$\sigma' = \sqrt{\left( \frac{32M}{\pi d^3} + \frac{4F}{\pi d^2} \right)^2 + 3\left( \frac{16T}{\pi d^3} \right)^2} \qquad \sigma' = \frac{S_y}{n}$$

$$\frac{S_y}{n} = \frac{32}{\pi d^3} \sqrt{\left( M + \frac{d}{8} F \right)^2 + \frac{3}{4} T^2}$$

In the presence of axial load F, diameter can not be solved in closed form.

In case F=0, diameter can be solved in closed form:

$$d = \left\{ \frac{32n}{\pi S_y} \sqrt{M^2 + \frac{3}{4} T^2} \right\}^{\frac{1}{3}}$$

# Fatigue Analysis of Shafts

- Shafts are subjected to a combination of loading modes.
- A common type of loading is fully reversed bending ($M_a$) and steady torsion ($T_m$).
- Rather than a unique design formula, a number of approaches exist.
- The general idea is as follows:

# Fatigue Analysis of Shafts

- Obtain mean and alternating equivalent stresses by using static design criteria.
  - Distortion energy theory - Von Mises Stress
  - Maximum shear stress theory - Maximum shear stress (on the critical plane !)
- By using these equivalent stresses apply a fatigue design criterion such as;
  - Soderberg
  - Goodman
  - Gerber, etc.

# Fatigue Analysis of Shafts

- Simplest approach is based on Von-Mises Stress.
- For a rather general case, there will be both mean and alternating shear and normal stresses but there is no axial load.
- Then stresses are;

$$\sigma_{xa} = K_f \frac{32M_a}{\pi d^3} \qquad \tau_{xya} = K_{fs} \frac{16T_a}{\pi d^3}$$

$$\sigma_{xm} = \frac{32M_m}{\pi d^3} \qquad \tau_{xym} = \frac{16T_m}{\pi d^3}$$

# Equivalent Stresses

$$\sigma' = \left\{ \frac{1}{2}\left[ (\sigma_x - \cancelto{0}{\sigma_y})^2 + (\cancelto{0}{\sigma_y} - \cancelto{0}{\sigma_z})^2 + (\cancelto{0}{\sigma_z} - \sigma_x)^2 \right] + 3(\tau_{xy}{}^2 + \cancelto{0}{\tau_{yz}}{}^2 + \cancelto{0}{\tau_{zx}}{}^2) \right\}^{\frac{1}{2}}$$

$$\sigma' = \sqrt{\sigma_x{}^2 + 3\tau_{xy}{}^2} \Longrightarrow \sigma'_a = \sqrt{\sigma_{xa}{}^2 + 3\tau_{xya}{}^2} \quad \& \quad \sigma'_m = \sqrt{\sigma_{xm}{}^2 + 3\tau_{xym}{}^2}$$

$$\sigma'_a = \sqrt{\left( K_f \frac{32M_a}{\pi d^3} \right)^2 + 3\left( K_{fs} \frac{16T_a}{\pi d^3} \right)^2} \qquad \sigma'_m = \sqrt{\left( \frac{32M_m}{\pi d^3} \right)^2 + 3\left( \frac{16T_m}{\pi d^3} \right)^2}$$

$$\sigma'_a = \frac{32}{\pi d^3} \sqrt{\left(K_f M_a\right)^2 + \frac{3}{4}\left(K_{fs} T_a\right)^2} \qquad \sigma'_m = \frac{32}{\pi d^3} \sqrt{M_m{}^2 + \frac{3}{4} T_m{}^2}$$

# Fatigue Design Criteria

- Let's use Soderberg Criteria: $\dfrac{\sigma_a'}{S_e} + \dfrac{\sigma_m'}{S_y} = \dfrac{1}{n}$

$$\frac{32}{S_e \pi d^3}\sqrt{\left(K_f M_a\right)^2 + \frac{3}{4}\left(K_{fs} T_a\right)^2} + \frac{32}{S_y \pi d^3}\sqrt{M_m{}^2 + \frac{3}{4} T_m{}^2} = \frac{1}{n}$$

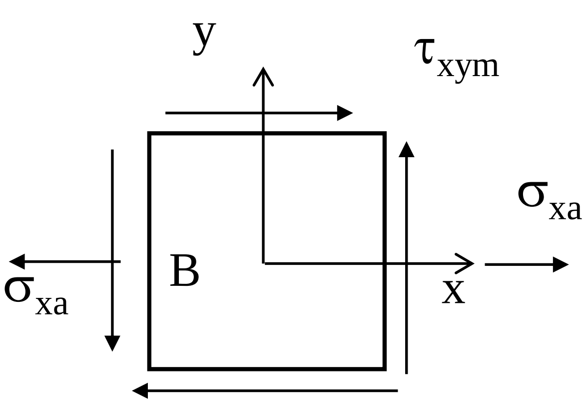
Solving $d$

$$d = \left\{ \frac{32n}{\pi} \left( \frac{1}{S_e}\sqrt{\left(K_f M_a\right)^2 + \frac{3}{4}\left(K_{fs} T_a\right)^2} + \frac{1}{S_y}\sqrt{M_m{}^2 + \frac{3}{4} T_m{}^2} \right) \right\}^{\frac{1}{3}}$$

If Goodman were used rather than Soderberg, $S_y$ would be replaced by $S_{ut}$ and yielding would be separately checked.

# Shaft design based on shear stress

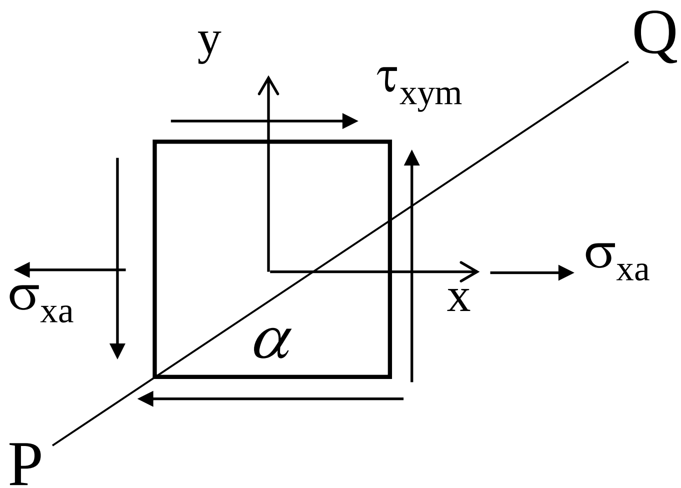
- Shaft design based on shear stress is a little bit more complicated.
- It involves finding the critical plane.
- As an example consider a case with only mean shear stress and alternating bending moment.

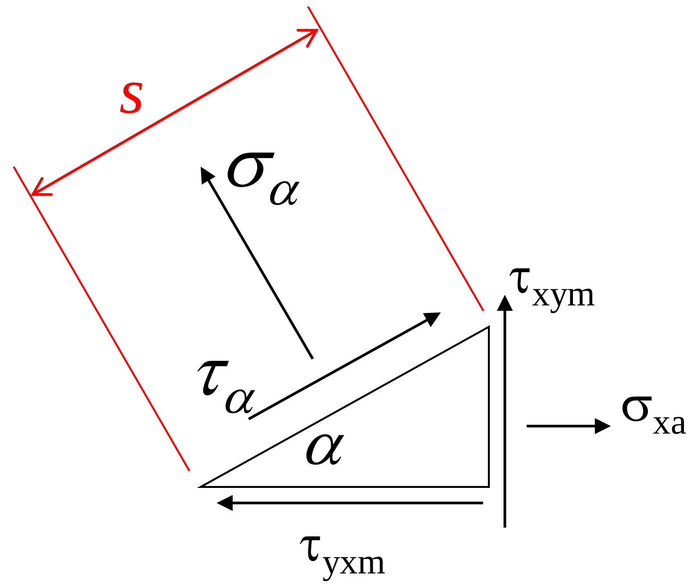



$$\sigma_x = \frac{32M_a}{\pi d^3}\cos\omega t \qquad \tau_{xym} = \frac{16T_m}{\pi d^3}$$

Angular speed

- Consider the plane PQ, making an angle $\alpha$ with the x-axis




- Since we shall use shear stress, we want to find $\tau_\alpha$ . (Subscripts *a* and *m* are dropped for convenience)

$$\sum F_{PQ} = 0$$

$$\tau_\alpha s - \left(\tau_{yx} s \cos\alpha\right)\cos\alpha + \left(\tau_{xy} s \sin\alpha\right)\sin\alpha + \left(\sigma_x s \sin\alpha\right)\cos\alpha = 0$$

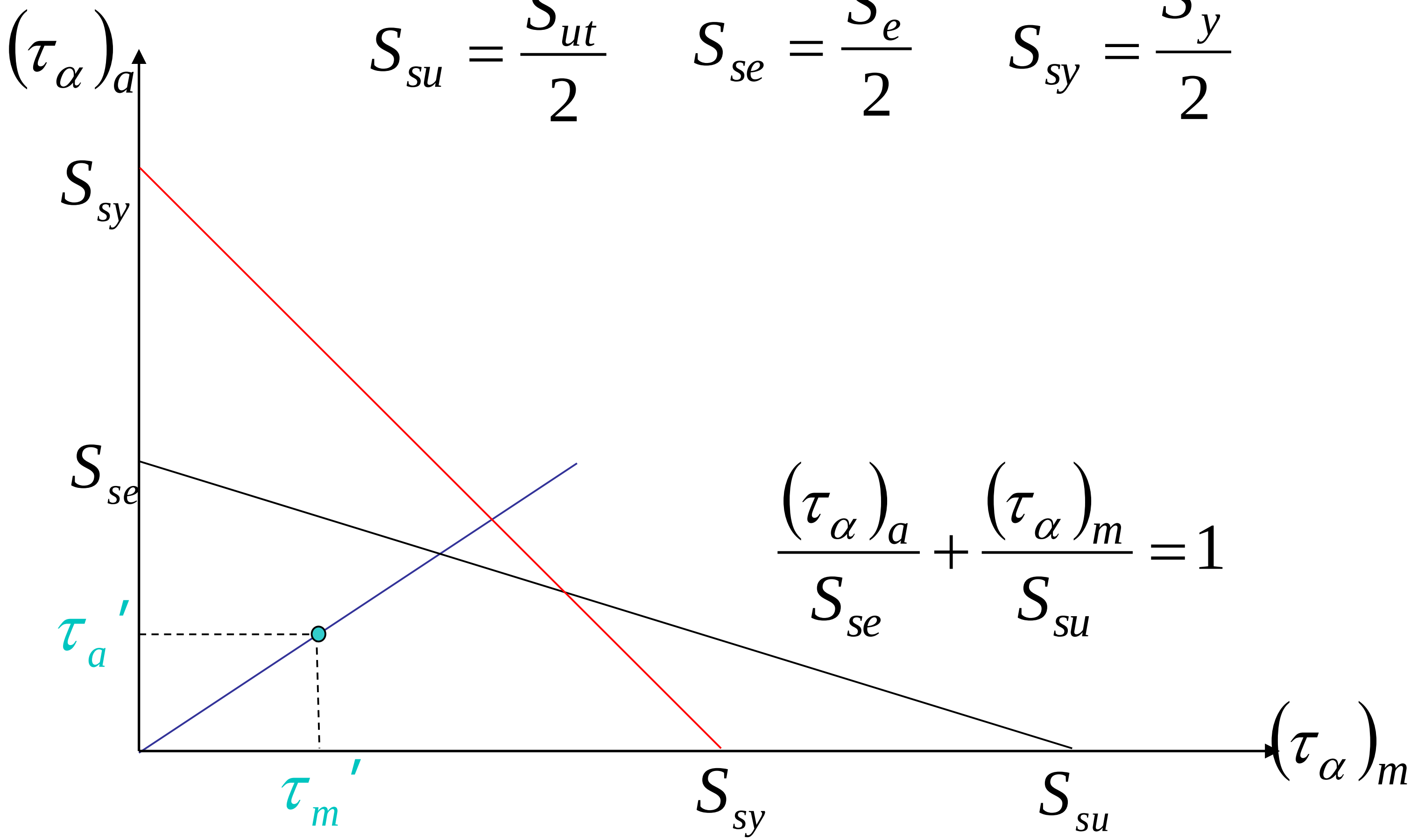
$$\tau_\alpha = \tau_{xy}\cos 2\alpha - \frac{\sigma_x}{2}\sin 2\alpha$$

$$\tau_\alpha = \underbrace{\frac{16T_m}{\pi d^3}\cos 2\alpha}_{(\tau_\alpha)_m} - \underbrace{\frac{16M_a}{\pi d^3}\sin 2\alpha}_{(\tau_\alpha)_a}\cos\omega t$$

- Note that mean and alternating components of $\tau_\alpha$ are functions of $\alpha$.
- For a certain $\alpha$, the combination of $((\tau_\alpha)_m,(\tau_\alpha)_a)$ will become most critical.
- At this point we should also invoke a fatigue design approach.

- Let's use modified Goodman for shear stresses.

$$S_{su} = \frac{S_{ut}}{2} \qquad S_{se} = \frac{S_e}{2} \qquad S_{sy} = \frac{S_y}{2}$$




$$\frac{(\tau_\alpha)_a}{S_{se}} + \frac{(\tau_\alpha)_m}{S_{su}} = 1$$

- Observe that $\cos 2\alpha = \dfrac{(\tau_\alpha)_m}{16T_m / \pi d^3}$ $\quad \sin 2\alpha = \dfrac{(\tau_\alpha)_a}{16M_a / \pi d^3}$

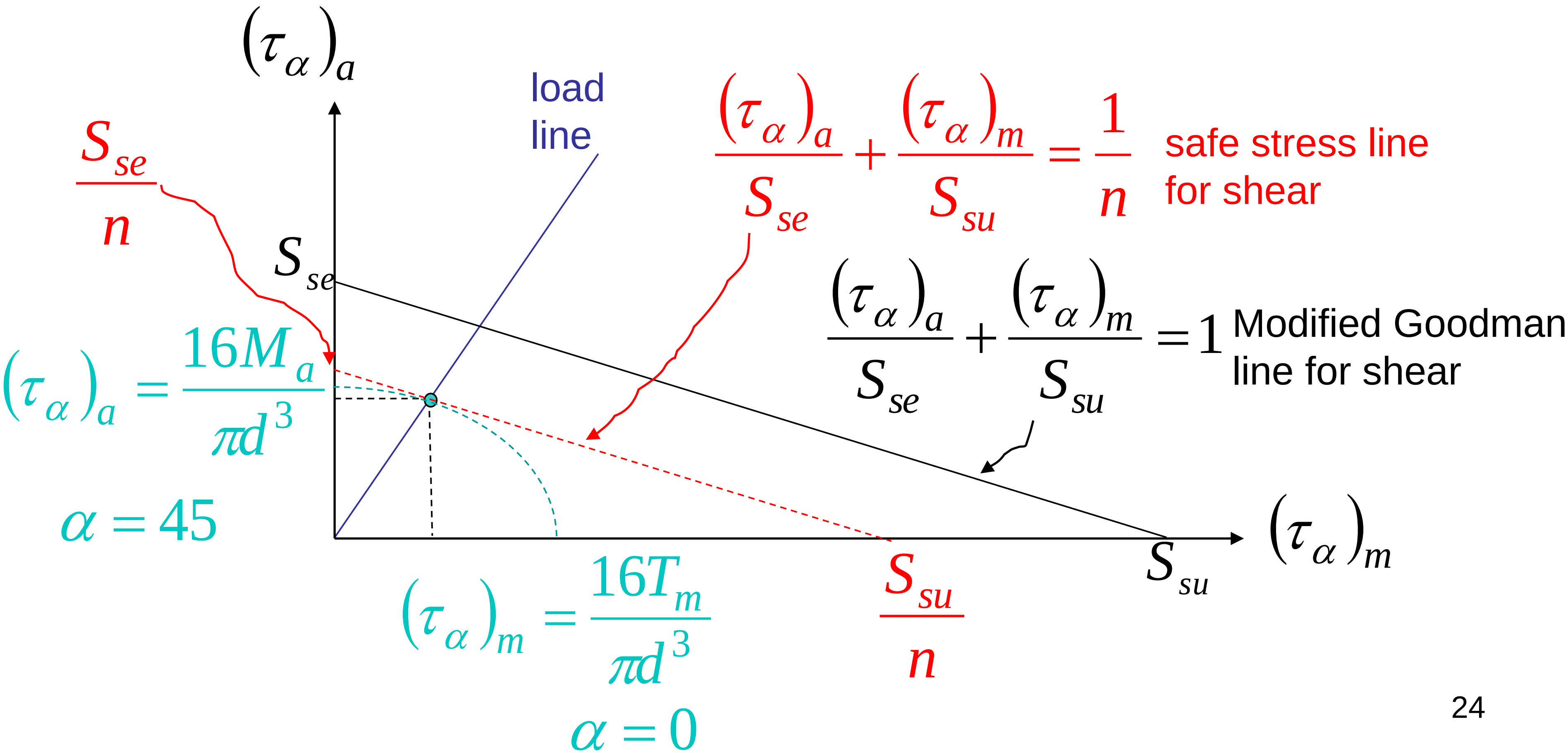
- But since $\cos^2 2\alpha + \sin^2 2\alpha = 1$ we get

$$\left( \frac{(\tau_\alpha)_m}{16T_m / \pi d^3} \right)^2 + \left( \frac{(\tau_\alpha)_a}{16M_a / \pi d^3} \right)^2 = 1$$

- This is the equation of an ellipse in the $((\tau_\alpha)_m, (\tau_\alpha)_a)$ plane, like

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \qquad \frac{dy}{dx} = -\frac{b^2}{a^2}\frac{x}{y}$$

- Plotting the ellipse on modified Goodman diagram,we note that each point on it corresponds to a certain $\alpha$.
- We can observe the most critical plane as the point closest to the Goodman line.



- The most critical point satisfies the equations of
  - the ellipse
  - the load line
  - the safe stress line
- Furthermore, the slope of the tangent to the ellipse at  the critical point is equal to the slope of safe stress and modified goodman lines.
- From the equation of ellipse;

$$\frac{d(\tau_\alpha)_a}{d(\tau_\alpha)_m} = -\frac{M_a}{T_m}\frac{1}{\tan 2\alpha}$$

- Equate this to slope of M. Goodman line to get the equation of $\alpha$ which corresponds to the critical plane.

$$(\tau_\alpha)_a = -\frac{S_{se}}{S_{su}}(\tau_\alpha)_m + 1 \qquad \text{(Goodman line)}$$

$$\frac{d(\tau_\alpha)_a}{d(\tau_\alpha)_m} = -\frac{M_a}{T_m}\frac{1}{\tan 2\alpha} = -\frac{S_{se}}{S_{su}} \Longrightarrow \alpha^* = \frac{1}{2}\arctan\frac{M_a}{T_m}\frac{S_{su}}{S_{se}}$$

- Equation of load line:

$$(\tau_\alpha)_a = \frac{16M_a/\pi d^3}{16T_m/\pi d^3}\frac{\sin 2\alpha^*}{\cos 2\alpha^*}(\tau_\alpha)_m \qquad (\tau_\alpha)_a = \frac{M_a}{T_m}\tan 2\alpha^*(\tau_\alpha)_m$$

- Substituting $\alpha^*$ in the equation of load line:

$$(\tau_\alpha)_a = \underbrace{\frac{{M_a}^2}{{T_m}^2}\frac{S_{su}}{S_{se}}}_{r}(\tau_\alpha)_m$$

- Substituting the equation above into
  – safe stress line, and
  – equation of ellipse

$$\frac{(\tau_\alpha)_a}{S_{se}} + \frac{(\tau_\alpha)_m}{S_{su}} = \frac{1}{n}$$

$$\left(\frac{(\tau_\alpha)_m}{16T_m/\pi d^3}\right)^2 + \left(\frac{(\tau_\alpha)_a}{16M_a/\pi d^3}\right)^2 = 1$$

$$(\tau_\alpha)_m = \frac{1}{n}\frac{1}{\frac{r}{S_{se}} + \frac{1}{S_{su}}} \quad (1)$$

$$(\tau_\alpha)_m{}^2 = \frac{1}{\left(\frac{1}{16T_m/\pi d^3}\right)^2 + \left(\frac{r}{16M_a/\pi d^3}\right)^2} \quad (2)$$

Equating square of eq. (1) to eq. (2),

- substituting definition of $r$,
- replacing $M_a$ with $K_f M_a$ to account for any possible stress concentration, and
- Replacing $S_{se}=S_e/2$, $S_{su}=S_{ut}/2$,

one can solve diameter $d$ as follows:

$$d = \left\{ \frac{32n}{\pi} \sqrt{\left( \frac{K_f M_a}{S_e} \right)^2 + \left( \frac{T_m}{S_{ut}} \right)^2} \right\}^{\frac{1}{3}}$$

- Accounting for presence of $M_m$ and $K_{fs}T_a$ , using Soderberg criterion and proceeding in the same manner one arrives at "Westinghouse Code Formula" which gives diameter $d$ as follows:

$$d = \left\{ \frac{32n}{\pi} \sqrt{\left( \frac{M_m}{S_y} + \frac{K_f M_a}{S_e} \right)^2 + \left( \frac{T_m}{S_y} + \frac{K_{fs} T_a}{S_e} \right)^2} \right\}^{\frac{1}{3}}$$

# Critical Speed of Shafts

- At certain speeds (which match resonant frequency of the sytem consisting of the shaft and the ensemble of attachments) the shaft becomes unstable, with deflections increasing without upper bound.
- These speeds are called critical speeds.
- The lowest critical speed can be easily estimated by Rayleigh Method.

# Critical Speed of Shafts

- According to our text book, we should seek first critical speeds at least twice the operating speed.
- In Rayleigh Method, the static deflection curve is used to estimate the lowest critical speed.
- Rayleigh’s equation overestimates the critical speed.

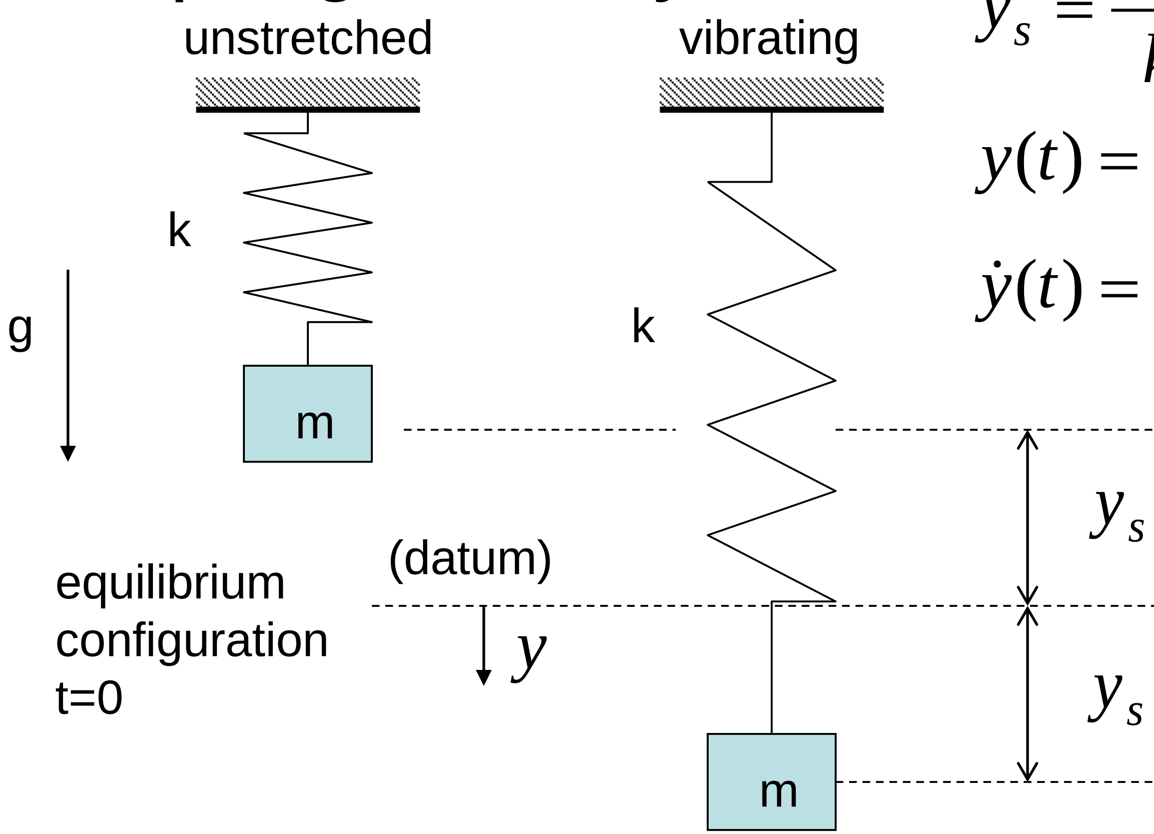
# Rayleigh's Equation

- First consider a single degree of freedom spring mass system.

$$y_s = \frac{mg}{k} \quad \text{static deflection}$$

$$y(t) = y_s \sin \omega t$$

$$\dot{y}(t) = y_s \omega \cos \omega t$$




$$(PE)_{\max} \quad (KE) = 0 \qquad (2)$$

$$(KE)_{\max} \quad (PE) = 0 \qquad (1)$$

$$(PE)_{\max} \quad (KE) = 0$$

$$\dot{y}_{\max} = y_s \omega$$

Conservation of Energy between (1) and (2)

$$(T_2 - T_1) + (V_{g2} - V_{g1}) + (V_{e2} - V_{e1}) = 0$$

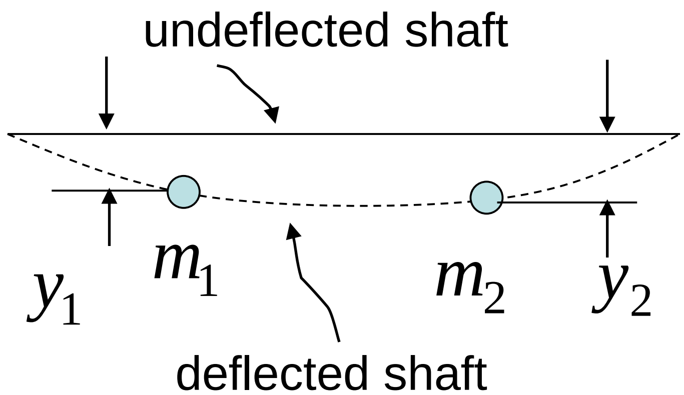
$$\left(0 - \frac{1}{2} m y_s{}^2 \omega^2\right) + (mgy_s - 0) + \underbrace{\left(0 - \frac{1}{2} k y_s{}^2\right)}_{-\frac{1}{2} mgy_s} = 0$$

But $k = \dfrac{mg}{y_s} \longrightarrow -\dfrac{1}{2} mgy_s$

$$\omega^2 \left(\frac{1}{2} m y_s{}^2\right) = \frac{1}{2} mgy_s \longrightarrow \omega^2 = \frac{\frac{1}{2} mgy_s}{\frac{1}{2} m y_s{}^2} = \frac{g}{y_s}$$

Rayleigh's method: $(KE)_{max}=(PE)_{max}$

# Generalization to shafts




$$PE = \frac{1}{2} g\left(m_1 y_1 + m_2 y_2 + \cdots\right)$$

$$KE = \frac{1}{2} \omega^2\left(m_1 {y_1}^2 + m_2 {y_2}^2 + \cdots\right)$$

Rayleigh's method: $(KE)_{max}=(PE)_{max}$

$$\omega^2 = g\,\frac{m_1 y_1 + m_2 y_2 + \cdots}{m_1 {y_1}^2 + m_2 {y_2}^2 + \cdots}$$

So the problem reduces to finding deflections at the locations of the masses.

In the case of overhanging shafts, the loads on the overhanging part should be reversed while finding deflections for a better approximation.